Progress in Nonlinear Differential Equations
and Their Applications

Etienne Sandier
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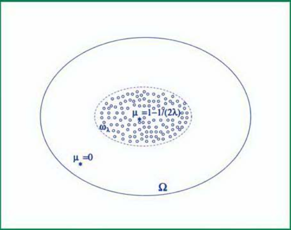
Vortices in the Magnetic Ginzburg–Landau Model



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Preface

More than ten years have passed since the book of F. Bethuel, H. Brezis and F. Hélein, which contributed largely to turning Ginzburg–Landau equations from a renowned physics model into a large PDE research field, with an ever-increasing number of papers and research directions (the number of published mathematics papers on the subject is certainly in the several hundreds, and that of physics papers in the thousands).

Having ourselves written a series of rather long and intricately interdependent papers, and having taught several graduate courses and mini-courses on the subject, we felt the need for a more unified and self-contained presentation.

The opportunity came at the timely moment when Haïm Brezis suggested we should write this book. We would like to express our gratitude towards him for this suggestion and for encouraging us all along the way.

As our writing progressed, we felt the need to simplify some proofs, improve some results, as well as pursue questions that arose naturally but that we had not previously addressed. We hope that we have achieved a little bit of the original goal: to give a unified presentation of our work with a mixture of both old and new results, and provide a source of reference for researchers and students in the field.

We are also grateful to all the colleagues who over the years have shared with us their knowledge and ideas on Ginzburg–Landau and on related topics, in particular: Fabrice Bethuel, Haïm Brezis, Frédéric Hélein, Tristan Rivière, Fanghua Lin, Peter Sternberg, Jacob Rubinstein, Bernard Helffer, Robert Jerrard, Mete Soner, Petru Mironescu, Robert Kohn, Amandine Aftalion, Lia Bronsard, Stan Alama, Frank Pacard, Raphaël Danchin, Stephen Gustafson, Daniel Spirn, Yaniv Almog, Luigi Ambrosio, Itai Shafrir, Didier Smets, Sisto Baldo, Giandomenico Orlandi, Patricia Bauman, and Dan Phillips.

This book would not have been possible without numerous visits to our respective institutions; we would thus like to thank the Courant Institute and the University of Paris XII for their hospitality, and acknowledge in particular support from Paris XII, the National Science Foundation and the Sloan foundation.

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Etienne Sandier
Sylvia Serfaty
October 2006

Chapter 1

Introduction

This book is devoted to the mathematical study of the two-dimensional Ginzburg–Landau model with magnetic field. This is a model of great importance and recognition in physics (with several Nobel prizes awarded for it: Landau, Ginzburg, and Abrikosov). It was introduced by Ginzburg and Landau (see [101]) in the 1950s as a phenomenological model to describe superconductivity. Superconductivity was itself discovered in 1911 by Kammerling Ohnes. It consists in the complete loss of resistivity of certain metals and alloys at very low temperatures. The two most striking consequences of it are the possibility of permanent *superconducting currents* and the particular behavior that, when the material is submitted to an external magnetic field, that field gets expelled from it. Aside from explaining these phenomena, and through the very influential work of A. Abrikosov [1], the Ginzburg–Landau model allows one to predict the possibility of a *mixed state* in type II superconductors where triangular vortex lattices appear. These vortices — in a few words a vortex can be described as a quantized amount of vorticity of the superconducting current localized near a point — have since been the objects of many observations and experiments. The first observation dates back from 1967, by Essman and Trauble, see [93]. For pictures of lattice observations in superconductors and more references to experimental results, refer to the web page `http://www.fys.uio.no/super/vortex/`.

The Ginzburg–Landau theory has also been justified as a limit of the Bardeen–Cooper–Schrieffer (BCS) quantum theory [29], which explains superconductivity by the existence of "Cooper pairs" of superconducting electrons.

In addition to its importance in the modelling of superconductivity, the Ginzburg–Landau model turns out to be the simplest case of a gauge theory, and vortices to be the simplest case of topological solitons (for these aspects see [138, 112, 194, 190] and the references therein); moreover, it is mathematically extremely close to the Gross–Pitaevskii model for superfluidity (see for example [191, 185]), and models for rotating Bose–Einstein condensates (see [2]), in which quantized vortices are also essential objects, and to which the Ginzburg–Landau techniques have been successfully exported.

1.1 The Model

After a series of reductions, which are described in Chapter 2, the 2D Ginzburg–Landau model leads to describing the state of the superconducting sample submitted to the external field h_{ex}, below the critical temperature, through its Gibbs energy:

$$G_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + |h - h_{\mathrm{ex}}|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}. \tag{1.1}$$

In this expression, Ω is a two-dimensional open subset of $\mathbb{R}^2$, which in our study is always assumed for simplicity to be smooth, bounded and simply connected. One can imagine that it represents the section of an infinitely long cylinder. Certain authors also use this as a simplified model for thin films.

The first unknown u is a *complex*-valued function, called an "order parameter" in physics, where it is generally denoted as ψ. It is a sort of "wave function", indicating the local state of the material or the phase in the Landau theory of phase transitions: $|u|^2$ is the density of Cooper pairs of superconducting electrons in the BCS approach. With our normalization, $|u| \le 1$ and where $|u| \simeq 1$ the material is in the superconducting phase, while where $|u| = 0$, it is in the normal phase (i.e., behaves like a normal conductor); the two phases are able to coexist in the sample.

The second unknown is A, the electromagnetic vector-potential of the magnetic-field, a function from Ω to $\mathbb{R}^2$. The magnetic field in the sample is deduced by $h = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$, it is thus a real-valued function in Ω. The notation ∇_A denotes the covariant gradient $\nabla - iA$; $\nabla_A u$ is thus a vector with complex components.

The *superconducting current* is a real vector given by $(iu, \nabla_A u)$ where $(\cdot, \cdot)$ denotes the scalar-product in $\mathbb{C}$ identified with $\mathbb{R}^2$. It may also be written as

$$\frac{i}{2}\left(u\overline{\nabla_A u} - \bar{u}\nabla_A u\right),$$

where the bar denotes the complex conjugation. The energy admits a *gauge-invariance*: it is invariant under the action of the unitary group $\mathbb{U}(1)$ in the form $u \to ue^{if}, A \to A + \nabla f$; we will come back to this in Chapters 2 and 3.

The parameter $h_{\mathrm{ex}} > 0$ represents the intensity of the applied field (assumed to be perpendicular to the plane of Ω). Finally, the parameter ε is the inverse of the "Ginzburg–Landau parameter" usually denoted κ, a non-dimensional parameter depending only on the material, ratio of the penetration depth (scale of variation of h) and the coherence length (scale of variation of u), also see Chapter 2. We will be interested in the regime of small ε, corresponding to large-κ (or extreme type-II) superconductors. The limit $\varepsilon \to 0$ or $\kappa \to \infty$ that we will consider is also called the London limit. In this limit, the characteristic size of the vortices, ε, tends to 0 and vortices become point-like.

The stationary states of the system are the critical points of G_ε, or the solutions of the Ginzburg–Landau equations:

$$(GL)\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = (iu, \nabla_A u) & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega \\ \nu \cdot \nabla_A u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nabla^\perp$ denotes the operator $(-\partial_2, \partial_1)$, and ν the outer unit normal to $\partial\Omega$. For more on the model and on the physics, we refer to Chapter 2 and the physics literature, in particular [192, 164, 80].

1.1.1 Vortices

We now need to more precisely explain a vortex. A vortex is an object centered at an isolated zero of u, around which the phase of u has a nonzero winding number, called the *degree of the vortex*, cf. Fig. 1.1 where vortices of degree 1 and -1 are represented.

When ε is small, it is clear from (1.1) that $|u|$ prefers to be close to 1, and a scaling argument hints that $|u|$ is different from 1 in regions of characteristic size ε. A typical behavior for u at a vortex of degree

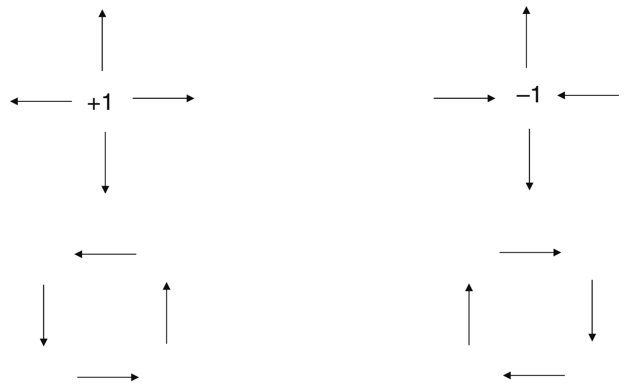

Figure 1.1: *Vortices of degree* $+1$ *and* -1*, at the arrows represent* u *in the complex plane, top and below they represent the current.*

d is $u(r, \theta) = f(r)e^{id\theta}$ in polar coordinates, with $f(0) = 0$. Of course this is an intuitive picture and several mathematical notions will be used to describe the vortices; one of our tasks will consist in relating these descriptions.

1.1.2 Critical Fields

Given ε, the behavior of minimizers and critical points of (1.1) is determined by the value of the external field h_{ex}.

There are three main critical values of h_{ex} or *critical fields* H_{c_1}, H_{c_2}, and H_{c_3}, for which phase-transitions occur. Below the first critical field, which is of order $O(|\log \varepsilon|)$ (as first established by Abrikosov), the superconductor is everywhere in its superconducting phase $|u| \sim 1$ and the magnetic field does not penetrate (this is called the Meissner effect or Meissner state). At H_{c_1}, the first vortice(s) appear. Between H_{c_1} and H_{c_2} the superconducting and normal phases (in the form of vortices) coexist in the sample, and the magnetic field penetrates through the vortices. This is called the *mixed state*, see for example Fig. 1.2. The larger $h_{\text{ex}} > H_{c_1}$ is, the more vortices there are. Since they repel each other, they tend to arrange in triangular Abrikosov lattices in order to minimize their repulsion. When $H_{c_2} \sim \frac{1}{\varepsilon^2}$, the vortices are so densely packed that they overlap each other, and at H_{c_2} a second phase transition occurs, after which $|u| \sim 0$ inside the sample, i.e., all superconductivity in the

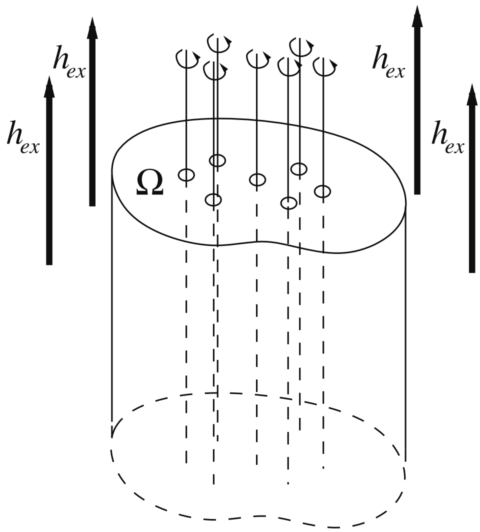

Figure 1.2: *Schematic representation of a superconducting cylinder with vortices.*

bulk of the sample is lost.

In the interval $[H_{c_2}, H_{c_3}]$ however, superconductivity persists near the boundary, this is called *surface superconductivity*, and after $H_{c_3} = O(\frac{1}{\varepsilon^2})$, superconductivity is completely destroyed and $u \equiv 0$, that is the sample is completely in the normal phase, the magnetic field completely penetrates and all superconductivity is lost (the phase transition really happens while decreasing the field below H_{c_3}).

For more on the critical fields and what results have been proved, we refer to Chapters 2 and 14.

1.2 Questions Addressed in this Book

Our goal is to describe, through rigorous mathematical analysis, in the asymptotic limit of ε small, the minimizers of (1.1) and their critical points in terms of their vortices. This comprises, in particular, determining their precise optimal vortex-locations. When the number of vortices becomes large (or blows up as $\varepsilon \to 0$), then, we describe the solutions through their vortex-densities (or "vorticity"). We give asymptotic expansions of the energy of solutions in terms of their vortices, and derive rigorously and with more precision the values of the critical fields which were known in the physics literature.

We deal with two aspects of the $\varepsilon \to 0$ limit. One is to establish the variational convergence of G_ε in all regimes of applied fields. Via energy-based methods, we are able to identify the Γ-limits of the energy, i.e., derive reduced problems, which can be solved, thus deducing the optimal limiting vortex repartitions for global minimizers. The second aspect is in passing to the limit as $\varepsilon \to 0$ in the Ginzburg–Landau equations (GL). This yields necessary stationarity conditions for a given measure to be an $\varepsilon \to 0$ limit of vorticity measures of critical points of G_ε.

1.3 Ginzburg–Landau with and without Magnetic Field: A Comparison

As we shall see in this book, the full Ginzburg–Landau energy G_ε is closely related to the simpler Ginzburg–Landau model without magnetic field:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.2}$$

In order to pass from one to the other, it suffices to set the magnetic potential A and the applied field h_{ex} to be zero in G_ε.

This model has been studied by numerous authors, after the pioneering work of Bethuel, Brezis, and Hélein in [43] (see also Chapter 14 for more details). The equation associated with (1.2) is

$$-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2). \tag{1.3}$$

It is a complex-valued version of the Allen–Cahn model for phase transitions (see [143]), leading to codimension 2, instead of codimension 1, singularities (the vortices).

The techniques developed for (1.1) follow the spirit of those developed for (1.2). Techniques and concepts were often first developed for the model without magnetic field, such as: renormalized energies, the Pohozaev identities, lower bounds for the energy in terms of the vortices, and stationarity conditions like in Theorem 1.7. In fact, the program is roughly the same for both energies, and the mathematical tools (presented here in Chapters 4 to 6) can be used for either energy, for that reason we will often present results for (1.1) and (1.2) in parallel.

The results concerning the local behavior and profile of solutions are also valid for both since at small scales the magnetic field (when not too large) has almost no effect in the equation, as we shall see in Proposition 3.12.

On the other hand, understanding the model with magnetic field raises the specific questions of understanding the influence of the applied field and determining the critical fields. Because large applied fields force large numbers of vortices, we need to be able to handle numbers of vortices which are *unbounded* as $\varepsilon \to 0$. This is a crucial difference between our analysis and the one originally developed for (1.2). This leads to developing specific techniques such as the vortex-ball construction, and the approach consisting in analyzing vortices through the averaged *vorticity measures*.

Moreover, the model without magnetic field does not exhibit all the phenomena observed in superconductors: first, vortices always repel each other but it is the presence of the magnetic field which confines them near the center of the domain, as seen for example in Theorem 1.3; second, the applied field induces phase transitions and selects the number of vortices. In contrast, minimizing E_ε without constraint leads to the natural Neumann boundary condition but to trivial minimizers. In order to induce vorticity, one has to either consider nonminimizing solutions (which are generally unstable — see Section 14.1.7) or to replace the effect of the applied field by a fixed Dirichlet boundary condition with nonzero degree as in [43]. However, this condition does not allow for unbounded numbers of vortices and hence for lattices of vortices. In fact, without specifying any boundary condition, if solutions of (1.3) have unbounded numbers of vortices as $\varepsilon \to 0$, as we shall see in Theorem 13.2, their limiting density is 0 in the domain (under some regularity assumption), vortices tend to go to the boundary to minimize their repulsion, thus ruling out the possibility of vortex lattices.

1.4 Plan of the Book

The book consists of three parts: the first part (Chapters 3 through 6) presents the essential tools developed to answer these questions, the second part (Chapters 7 through 12) presents results obtained through minimization (Γ-convergence type results), and the third part (Chapter 13) contains results for nonminimizing solutions.

Let us now briefly describe our main results (more information is given in each chapter). The focus of the book is the limit as $\varepsilon \to 0$ and throughout, the notation $a \sim b$ will mean $\lim_{\varepsilon\to 0} a/b = 1$, and $a \ll b$ will mean $\lim_{\varepsilon\to 0} a/b = 0$.

1.4.1 Essential Tools

The book starts in Chapter 2 with a heuristic presentation of the model and of the phase diagrams (critical fields) for type-II superconductors, aimed at nonspecialists, and almost completely independent from the rest of the book.

Chapter 3 gathers basic mathematical results on the Ginzburg–Landau equation (existence of solutions, a priori estimates, particular solutions).

After these two introductory chapters come a series of chapters presenting the essential mathematical tools, which are used in all the remaining chapters.

Chapter 4 presents what is now known as the "ball-construction method". It is a method introduced independently by Jerrard [113] and Sandier [166], which allows one to obtain universal lower bounds for Ginzburg–Landau energies (either (1.1) or (1.2)) in terms of the vortices and their degrees, with possibly *unbounded numbers of vortices*, through a ball-growth method. Here we present an improved version of the estimate which can be phrased in the following way:

Theorem 1.1. *For any $\alpha \in (0,1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if (u, A) is a configuration such that $E_\varepsilon(|u|) \le \varepsilon^{\alpha-1}$, then for any $r \in (\varepsilon^{\frac{\alpha}{2}}, 1)$, there exists a finite collection of disjoint closed balls $\{B_i\}_i$ of the sum of the radii r, covering $\{|u| \le 1 - \varepsilon^{\frac{\alpha}{4}}\} \cap \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \ge \varepsilon\}$ such that*

$$\frac{1}{2}\int_{\cup_i B_i} |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \ge \pi D\left(\log \frac{r}{D\varepsilon} - C\right) \tag{1.4}$$

where $D = \sum_i |d_i|$, $d_i = \deg(u, \partial B_i)$, and C is a universal constant.

In this way, the balls we construct have small radii (the parameter of choice r), and, whatever definition we take of the vortex-region, they cover it. Moreover, we bound from below the energy contained in the vortex balls in terms of the degrees on the boundary of the balls, i.e., the degrees of the vortices, which is consistent with the known fact that vortices of degree d_i cost at least an order $\pi|d_i|\log\frac{1}{\varepsilon}$ of energy. The estimate (1.4) is slightly different due to possibly large numbers of vortices which can get very close to one another, but it is optimal as stated. Observe that this lower bound is very general, it does not require any hypothesis on (u, A) other than a reasonable (but quite large) upper bound on its energy.

In Chapter 5, we present an application of the ball-growth method, which can be read independently. It consists in coupling the ball-growth method with an energy-estimate obtained through the "Pohozaev identity". This coupling provides different lower bounds for the energy in terms of the potential term of the type

$$G_\varepsilon(u, A) \geq C \int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} |\log \varepsilon|.$$

In this case the estimates are not for arbitrary maps but for *solutions* of the Ginzburg–Landau equations (GL) or (1.3). We give applications of these estimates in describing the fine behavior of solutions of (GL).

In Chapter 6 we present another crucial tool that has been widely used in the literature for Ginzburg–Landau in any dimension since the work of Jerrard and Soner [119]: the Jacobian estimate. This estimate allows one to relate the vorticity measures $2\pi\sum_i d_i\delta_{a_i}$, naturally derived from the ball-construction method (here a_i are the centers of the balls, d_i's their degrees, and δ the Dirac mass), to a quantity which is more intrinsic to u: a gauge-invariant version of the Jacobian determinant of u

$$\mu(u, A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A$$

or $\operatorname{curl}(iu, \nabla u)$ without magnetic field. This is really the intrinsic vorticity quantity associated with (u, A) (exactly like the vorticity in fluid mechanics). The result expresses that if the balls are constructed not too large (as measured by r), then these two quantities are close in a weak norm:

Theorem 1.2. *Under the hypotheses of the previous theorem, for any* $\beta \in (0, 1)$*, we have*

$$\left\| \mu(u, A) - 2\pi \sum_i d_i \delta_{a_i} \right\|_{C_0^{0,\beta}(\Omega)^*} \leq r^\beta {G_\varepsilon}^0(u, A), \tag{1.5}$$

where ${G_\varepsilon}^0$ *is the energy when* $h_{ex} = 0$.

The previous theorem allowed for a control on the mass of $2\pi \sum_i d_i \delta_{a_i}$ as measures. Combining these two results yields compactness results on the vorticities $\mu(u, A)$. The relation (1.5), in which the right-hand side term is usually small, allows one to control the error between $\mu(u, A)$ and a density of vortices, and ensures that the limiting vorticities are measures.

1.4.2 Minimization Results

Assuming the main a priori bounds of Chapter 3 and the results of Theorems 1.1 and 1.2, the reader may skip to the more concrete applications of these results, beginning in Chapter 7.

With Chapter 7, we start to give results on the minimization of (1.1). This chapter contains the main Γ-convergence (in the sense of De Giorgi) result for G_ε expressed in terms of

$$\lambda = \lim_{\varepsilon \to 0} \frac{h_{\rm ex}}{|\log \varepsilon|}.$$

Theorem 1.3. *As* $\varepsilon \to 0$, $\frac{G_\varepsilon}{{h_{ex}}^2}$ Γ*-converges to*

$$E_\lambda(\mu) = \frac{\|\mu\|}{2\lambda} + \frac{1}{2} \int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2,$$

defined over bounded Radon measures in $H^{-1}(\Omega)$*, where* $\|\mu\|$ *is the total mass of* μ *and*

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu \text{ in } \Omega \\ \qquad\qquad h_\mu = 1 \text{ on } \partial\Omega. \end{cases}$$

The meaning of Γ-convergence is specified in Chapter 7, the most important fact being that for $(u_\varepsilon, A_\varepsilon)$ minimizing G_ε, the rescaled vorticities $\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}}$ converge in $C_0^{0,\beta}(\Omega)^*$ to a limiting measure μ_* which minimizes E_λ with $\frac{\min G_\varepsilon}{{h_{\rm ex}}^2} \to \min E_\lambda$, implying also that $\frac{h}{h_{\rm ex}} \to h_{\mu_*}$. E_λ has a unique minimizer, and it turns out that it can be identified through the solution of an obstacle problem

$$\min_{\substack{h - 1 \in H_0^1(\Omega) \\ h \geq 1 - \frac{1}{2\lambda}}} \frac{1}{2} \int_\Omega |\nabla h|^2 + h^2$$

by the fact that h_{μ_*} is the minimizer of the above problem.

Thus the limiting measure μ_* is determined by λ, and existing knowledge on the obstacle problem (which is a particular case of a free-boundary problem) tells us that it is a uniform measure supported in a subdomain ω_λ of Ω, see Fig. 1.3. Moreover, there exists a critical value

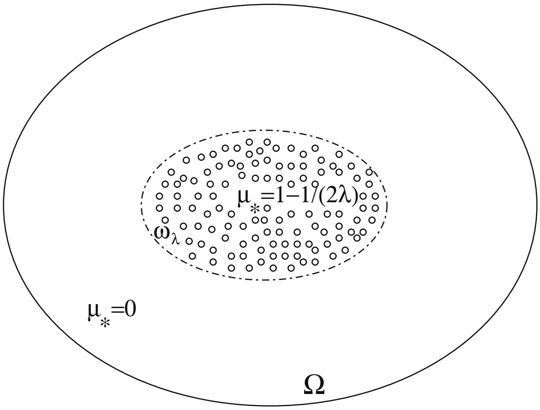

Figure 1.3: Optimal density of vortices according to the obstacle problem.

$C(\Omega)$ such that ω_λ is empty when $\lambda < C(\Omega)$ and ω_λ has positive measure if $\lambda > C(\Omega)$, hence in this case $\mu_* \neq 0$. When $\lambda = C(\Omega)$ the set ω_λ is finite—we denote it by Λ—hence the measure μ_* is zero in this case since it is the restriction of the Lebesgue measure to a set of measure zero.

Both Λ and $C(\Omega)$ are defined in terms of the solution h_0 to

$$\begin{cases} -\Delta h_0 + h_0 = 0 \text{ in } \Omega \\ \qquad\quad h_0 = 1 \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

Λ is the set where h_0 achieves its minimum and

$$C(\Omega) = \frac{1}{2\max_\Omega |h_0 - 1|}. \tag{1.7}$$

Starting from $\lambda = C(\Omega)$ and increasing λ, the set ω_λ grows, first around the points of Λ, and $\omega_\lambda \to \Omega$ as $\lambda \to \infty$.

Fixing $\varepsilon > 0$, the first critical field $H_{c_1}(\varepsilon)$ is usually defined by the fact that for $h_{\text{ex}} < H_{c_1}$ minimizers of the Ginzburg–Landau functional do not have vortices, while they do if $h_{\text{ex}} > H_{c_1}$, even though the existence of such a value for every $\varepsilon > 0$ remains to be proved. Theorem 1.3 and the above remarks tell us instead that

$$H^0_{c_1} = C(\Omega)|\log \varepsilon|$$

is an asymptotic critical value for h_{ex} in the sense that according to whether $(h_{\text{ex}}(\varepsilon) - H^0_{c_1})/|\log \varepsilon|$ tends to a negative (resp. positive) number, the limiting vorticity measure is zero (resp. nonzero), meaning that for small ε the number of vortices is negligible (resp. not negligible) compared to h_{ex}. In Chapter 12, Theorem 12.1, we will see that if $H_{c_1}(\varepsilon)$ is defined as above, we have $H_{c_1} \sim H^0_{c_1}$ as $\varepsilon \to 0$ (see also (1.8) below), with an explicit expansion up to $o(1)$.

In Chapter 8, we extend this study to higher applied fields such that $|\log \varepsilon| \ll h_{\text{ex}} \ll \frac{1}{\varepsilon^2}$, i.e., almost up to H_{c_2}. We show that in this situation the energy-minimization problem becomes local and can be solved by blowing up and using the result of Theorem 1.3. The energy-density and the vortex repartition are thus found to be uniform, as seen in:

Theorem 1.4. *Assume, as $\varepsilon \to 0$, that $|\log \varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$. Then, letting $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, and letting $g_\varepsilon(u, A)$ denote the energy-density $\frac{1}{2}\left(|\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$, we have*

$$\frac{2g_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{ex} \log \frac{1}{\varepsilon\sqrt{h_{ex}}}} \rightharpoonup dx \quad \text{as } \varepsilon \to 0$$

in the weak sense of measures, where dx denotes the two-dimensional Lebesgue measure; and

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u, A) \sim \frac{|\Omega|}{2} h_{ex} \log \frac{1}{\varepsilon\sqrt{h_{ex}}} \quad \text{as } \varepsilon \to 0,$$

where $|\Omega|$ is the area of Ω. Moreover

$$\frac{h_\varepsilon}{h_{ex}} \to 1 \quad \text{in } H^1(\Omega)$$

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \to dx \quad \text{in } H^{-1}(\Omega).$$

In both Theorems 1.3 and 1.4 we find an optimal limiting density which is constant on its support. This provides a first (but very incomplete) confirmation of the Abrikosov lattices of vortices observed and predicted in physics (see Chapter 2).

In Chapter 9, probably the most technical of all, we refine the study around the value $h_{\rm ex} \sim H^0_{c_1}$, assuming for simplicity that Λ is reduced to a single point p. In this regime, vortices concentrate around the point p, the limiting vortex density is 0 if rescaled by $h_{\rm ex}$ but not if suitably rescaled by the actual number of vortices n_ε. We study the intermediate regime where $1 \ll n_\varepsilon \ll h_{\rm ex}$, which requires very precise estimates (since it combines the difficulties of the unbounded number of vortices, and the ones of relatively small numbers of vortices). We again derive a Γ-convergence result and a limiting energy in this case, under the assumption that Λ is reduced to one point p: $G_\varepsilon/n_\varepsilon^2$ Γ-converges to

$$I(\mu) = -\pi \int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\, d\mu(x)\, d\mu(y) + \pi \int_{\mathbb{R}^2} Q(x)\, d\mu(x)$$

defined over the set of probability measures on $\mathbb{R}^2$, and where Q is a positive definite quadratic function (the Hessian of h_0 at p). In what follows, for any measure μ, $\tilde{\mu}$ denotes the push-forward of μ under the rescaling $x \mapsto \sqrt{\frac{h_{\rm ex}}{n_\varepsilon}}(x-p)$. Also, $f_\varepsilon(n)$ denotes an explicit quantity depending only on n, $h_{\rm ex}$, ε and Ω.

Theorem 1.5. *Assuming $\Lambda = \{p\}$, let $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be a family of configurations such that $G_\varepsilon(u_\varepsilon, A_\varepsilon) < \varepsilon^{-1/4}$ with $h_{ex} < C|\log\varepsilon|$. Defining n_ε as $\sum_i |d_i|$ where the d_i's are the degrees of some collection of vortex-balls of total radius $r = \frac{1}{\sqrt{h_{ex}}}$ constructed by Theorem 1.1, assume that*

$$1 \ll n_\varepsilon \ll h_{ex}$$

and $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le f_\varepsilon(n_\varepsilon) + Cn_\varepsilon^2$, as $\varepsilon \to 0$. Then there exists a probability measure μ_* such that, after extraction of a subsequence, $\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon} \to \mu_*$ in $(C_c^{0,\gamma}(\mathbb{R}^2))^*$ for each $\gamma > 0$ and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) - f_\varepsilon(n_\varepsilon) \ge n_\varepsilon^2 I(\mu_*) + o(n_\varepsilon^2).$$

Conversely, for each probability measure μ with compact support in $\mathbb{R}^2$ and each $1 \ll n_\varepsilon \ll h_{ex} \le C|\log\varepsilon|$, there exists $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that $\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon} \to \mu_*$ in $(C_c^{0,\gamma}(\mathbb{R}^2))^*$ for each $\gamma > 0$ and such that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) - f_\varepsilon(n_\varepsilon) \le n_\varepsilon^2 I(\mu) + o(n_\varepsilon^2).$$

As a consequence, in the regime $\log|\log\varepsilon| \ll h_{\rm ex} - H^0_{c_1} \ll |\log\varepsilon|$ we are able to determine to leading order the number $1 \ll n_\varepsilon \ll h_{\rm ex}$ of vortices of the minimizers of G_ε as a function of $h_{\rm ex}$ (see below) and to show that the limiting optimal vortex repartition (after rescaling/division by n) is μ_0, the unique minimizer of I, also see Fig. 1.8 below.

Chapter 10 is a preparation for Chapter 11 for dealing with the difficulties of bounded numbers of vortices (these are similar to those of [43] except that the vortices can get very close to one another). In Chapter 11 and the following, we complete the picture for minimizers of the energy by dealing with the regime $h_{\rm ex} - H^0_{c_1} = O(\log|\log\varepsilon|)$. In this case we prove the optimal number of vortices is bounded, and their limits are simply limiting vortex-points.

We characterize, again through a limiting (discrete) energy, the most favorable vortex-locations. After blow-up around p by the factor $\sqrt{\frac{h_{\rm ex}}{n}}$ (see Fig. 1.8), the vortices converge to a minimizer of the following limiting energy:

$$w_n(x_1,\dots,x_n) = -\pi\sum_{i\neq j}\log|x_i - x_j| + \pi n\sum_{i=1}^n Q(x_i),$$

a discrete version of I, reminiscent of the "renormalized energy" of [43]. When $Q(x) = C|x|^2$, the minimization of w_n has been studied by Gueron–Shafrir in [105] — replacing the term $\sum_{i=1}^n Q(x_i)$ by the constraint $\sum_{i=1}^n |x_i|^2 = 1$. Their theoretical and numerical results indicate that for $n \le 3$, the minimizers are regular polygons centered at the origin; for $7 \le n \le 10$ they are regular stars (= regular polygon + center); for $4 \le n \le 6$ both are locally minimizing and can be numerically obtained. In Figs. 1.4 and 1.5 we reproduce some of the shapes of minimizers obtained in their numerical simulations for higher n's.

These shapes are quite close to those observed in rotating superfluid helium (see [195, 191]) which, as we mentioned, is described through a similar model.

For all cases of $h_{\rm ex} - H_{c_1} \ll |\log\varepsilon|$, the optimal number of vortices for minimizers is given as follows: we exhibit an increasing sequence H_n of values of $h_{\rm ex}$ such that if $h_{\rm ex} \in [H_n, H_{n+1})$, the optimal number is n, and we show that

$$H_n \sim C(\Omega)|\log\varepsilon| + (n-1)C(\Omega)\log\frac{|\log\varepsilon|}{n} + \text{ lower order terms} \quad (1.8)$$

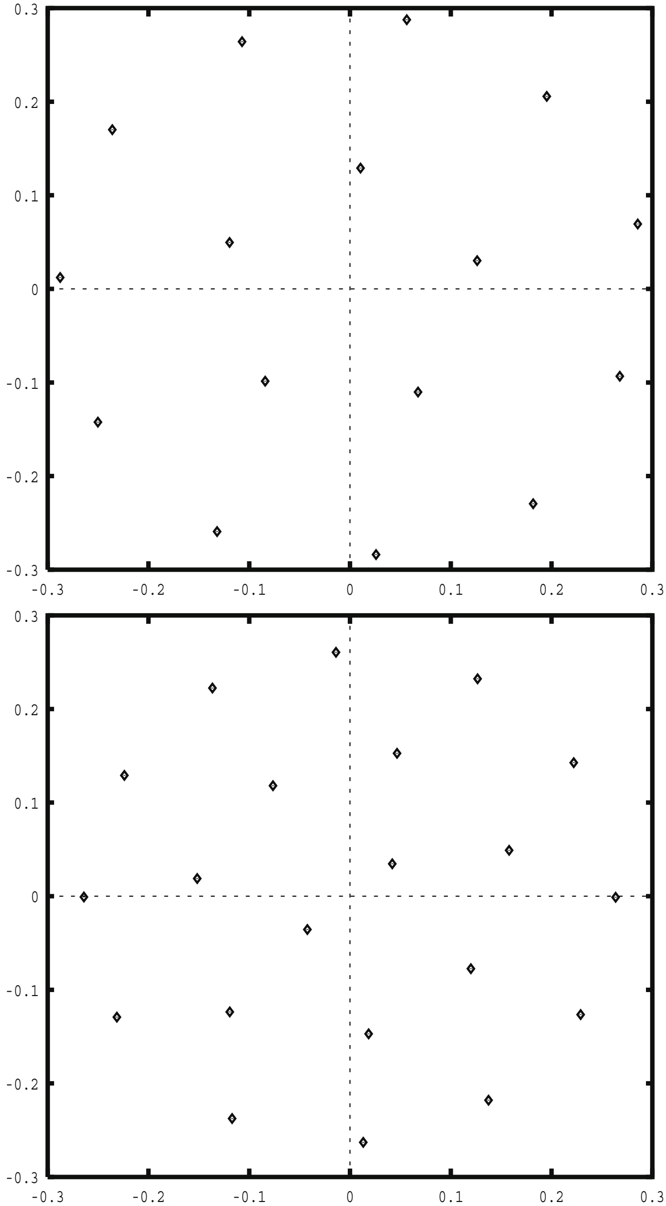

Figure 1.4: Results of the numerical optimization of [105] for w_n, $n = 16$ (*top*) and $n = 21$ (*bottom*).

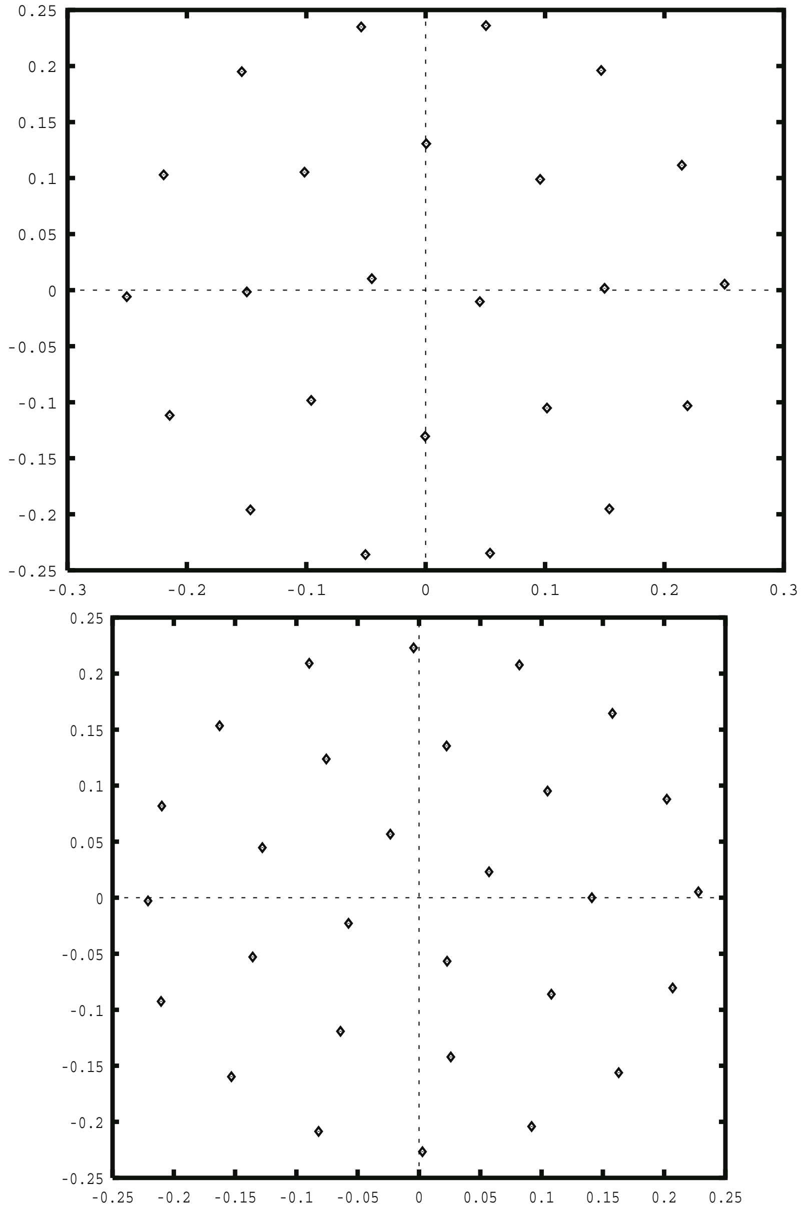

Figure 1.5: Results of the numerical optimization of [105] for w_n, $n = 24$ (*top*) and $n = 29$ (*bottom*).

(see (1.7), and (9.88), (12.2) for the precise formulae with expansions up to $o(1)$). These can be considered as successive critical fields $H_1 = H_{c_1}$, H_2, H_3, ... at which an additional nth vortex appears in minimizing configurations. The number of vortices found in minimizers increases rapidly after H_{c_1}, and more and more rapidly until it becomes $\sim h_{\rm ex}$ when $h_{\rm ex} \gg |\log \varepsilon|$ (see Fig. 1.6).

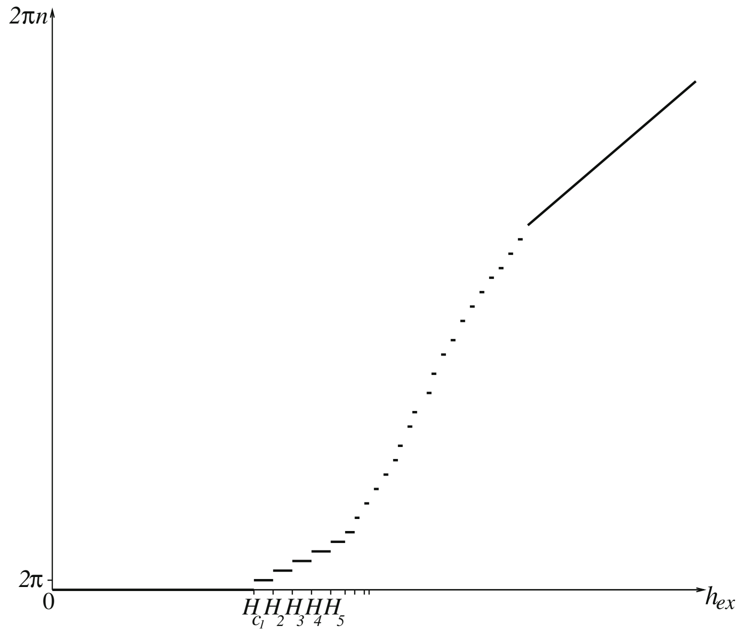

Figure 1.6: Schematic representation of the optimal number of vortices n with respect to $h_{\rm ex}$.

1.4.3 Branches of Local Minimizers

While we describe energy-minimizers for relatively small n's, we construct local minimizers of the energy which have prescribed numbers of vortices n. This solves an inverse-type problem: given a minimizer of w_n, show that there exist stable solutions of Ginzburg–Landau with n vortices of degree one, converging after blow-up to the minimizer of w_n. These solutions are obtained by a local minimization procedure: we minimize the energy over suitable subsets of the functional space. This corresponds, roughly speaking, to a way of minimizing the energy over configurations with a prescribed number n of vortices.

This construction is possible for n bounded, or n unbounded but not too large, and for a wide range of $h_{\rm ex}$.

Theorem 1.6. *For Ω as above and for any n and h_{ex} belonging to appropriate intervals, there exists ε_0 such that for any $\varepsilon < \varepsilon_0$, there exists a locally minimizing critical point $(u_\varepsilon, A_\varepsilon)$ of G_ε such that u_ε has exactly n zeroes $a_1^\varepsilon, \dots, a_n^\varepsilon$ and there exists $R > 0$ such that $|u_\varepsilon| \geq \frac{1}{2}$ in $\Omega \backslash \cup_i B(a_i^\varepsilon, R\varepsilon)$, with $deg(u_\varepsilon, \partial B(a_i^\varepsilon, R)) = 1$. Moreover,*

1. *If n and h_{ex} are constants independent of ε, up to extraction of a subsequence, the configuration $(a_1^\varepsilon, \dots, a_n^\varepsilon)$ converges as $\varepsilon \to 0$ to a minimizer of the function*

$$R_{n,h_{ex}} = -\pi \sum_{i \neq j} \log |x_i - x_j| + \pi \sum_{i,j} S_\Omega(x_i, x_j) + 2\pi h_{ex} \sum_{i=1}^{n} (h_0 - 1)(x_i).$$

   where S_Ω is the regular part of a Green's function associated with Ω.

2. *If n is independent of ε and $h_{ex} \to \infty$, up to extraction of a subsequence, the configuration of the $\tilde{a}_i^\varepsilon = \sqrt{\frac{h_{ex}}{n}}(a_i^\varepsilon - p)$ converges as $\varepsilon \to 0$ to a minimizer of w_n.*

3. *If $n_\varepsilon \to \infty$ and $h_{ex} \to \infty$, then again denoting $\tilde{a}_i^\varepsilon = \sqrt{\frac{h_{ex}}{n_\varepsilon}}(a_i^\varepsilon - p)$,*

$$\frac{1}{n_\varepsilon} \sum_{i=1}^{n_\varepsilon} \delta_{\tilde{a}_i^\varepsilon} \rightharpoonup \mu_0,$$

   the unique minimizer of I.

Thus, we have shown the multiplicity of stable solutions coexisting for a given $h_{\rm ex}$. We also have the explicit expression of the energy along these branches, so that we can determine among them, the energy-minimizing one is the one with n vortices, if $h_{\rm ex}$ is in the interval $[H_n, H_{n+1})$; but the other ones, being stable, can still be observed, see Fig. 1.7.

The lower (resp. upper) bound of the interval of values of $h_{\rm ex}$ over (resp. below) which a given branch of solutions is linearly stable is usually referred to as the subcooling (resp. superheating) field. Let us more precisely state Theorem 1.6 when n is independent of ε: the branch of locally minimizing solutions with n vortices exists for any ε small enough (depending on n) if h_{ex} is in a range $[c_n, \varepsilon^{-\alpha_n}]$, where c_n and α_n are independent of ε. Thus we get estimates for the subcooling and superheating fields of the branch of solutions with n vortices as $\varepsilon \to 0$, since these solutions are locally minimizing, hence stable. These estimates are probably not optimal: for instance the superheating field when $n = 1$ is expected to be of order $1/\varepsilon$, but we are not able to prove that the branch with one vortex exists for such large values of h_{ex}. Also note that we do not prove that our n vortex solutions depend smoothly on the parameter h_{ex}, which is often implied when speaking of a branch of solutions. We believe however that this is the case.

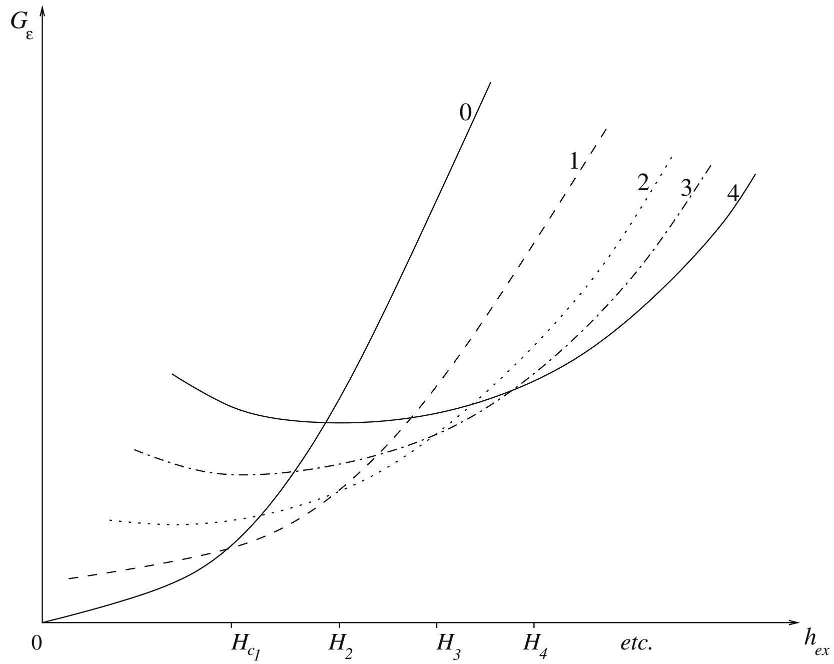

Figure 1.7: Branches of solutions with $n = 0$ vortex (Meissner solution), $n = 1$ vortex, $n = 2, 3, 4$ vortices ... with their energy.

Let us also point out that we have derived a series of limiting energies: E_λ, I, w_n, $R_{n,h_{\text{ex}}}$, each of them corresponding to a different regime in (n, h_{ex}): $R_{n,h_{\text{ex}}}$ for both n and h_{ex} bounded, w_n for n bounded and

$h_{\rm ex} \to \infty$, I for $1 \ll n \ll h_{\rm ex}$ and E_λ for $1 \ll n \sim Ch_{\rm ex}$, sort of limits of each other as summed up in the following chart:

$$\begin{array}{ccccc} I & \xleftarrow[n \ll h_{\rm ex}]{n \to \infty} & R_{n,h_{\rm ex}} & \xrightarrow[n \sim Ch_{\rm ex}]{n \to \infty} & E_\lambda \\ & {\scriptstyle n \to \infty} \nwarrow & \downarrow {\scriptstyle h_{\rm ex} \to \infty} & & \\ & & w_n & & \end{array}$$

Fig. 1.8 below is a rough picture of the vortices in such cases.

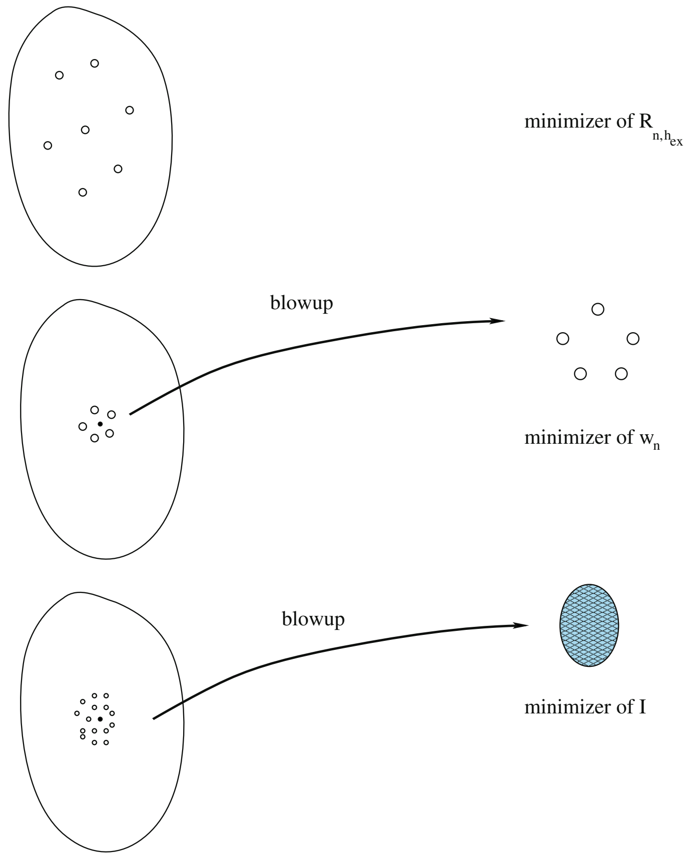

Figure 1.8: Schematic picture of the vortices for minimizers in the cases n and $h_{\rm ex}$ bounded, n bounded and $h_{\rm ex} \to +\infty$, and $1 \ll n \ll h_{\rm ex}$ respectively.

1.4.4 Results on Critical Points

Chapter 13 can be read independently from Chapters 7 to 12: it gives necessary conditions on limiting vorticities for arbitrary solutions of the Ginzburg–Landau equations, stable or unstable. It is a way of passing to the limit $\varepsilon \to 0$ in the Ginzburg–Landau equations, and to get a criticality condition on the limiting vorticities. The method consists in passing to the limit in the relation on the "stress-energy tensor" being divergence-free, i.e., in the conservative form of the Ginzburg–Landau equations.

Theorem 1.7. *Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be solutions of the Ginzburg–Landau equations such that $G_\varepsilon^0(u_\varepsilon, A_\varepsilon) \le C\varepsilon^{-\alpha}$ with $\alpha < 1/3$. Then for any $\varepsilon > 0$, there exists a measure ν_ε of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ where the sum is finite, $a_i^\varepsilon \in \Omega$ and $d_i^\varepsilon \in \mathbb{Z}$ for every i, such that letting $n_\varepsilon = \sum_i |d_i^\varepsilon|$,*

$$n_\varepsilon \le C\frac{G_\varepsilon^0(u_\varepsilon, A_\varepsilon)}{|\log\varepsilon|},$$

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{W^{-1,p}(\Omega)}\|\mu_\varepsilon - \nu_\varepsilon\|_{C^0(\Omega)^*} \to 0, \tag{1.9}$$

for some $p \in (1, 2)$.

Moreover, if $\{\nu_\varepsilon\}_\varepsilon$ are any measures satisfying (1.9) and n_ε is defined as above, then, possibly after extraction, one of the following holds.

0. *$n_\varepsilon = 0$ for every ε small enough and then μ_ε tends to 0 in $W^{-1,p}(\Omega)$.*

1. *$n_\varepsilon = o(h_{ex})$, and then, for some $p \in (1, 2)$, $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ such that*

$$\mu\nabla h_0 = 0, \tag{1.10}$$

   hence μ is a linear combination of Dirac masses supported in the finite set of critical points of h_0.

2. *$h_{ex} \sim \lambda n_\varepsilon$, with $\lambda > 0$, then for some $p \in (1, 2)$, μ_ε/h_{ex} converges in $W^{-1,p}(\Omega)$ to a measure μ and h_ε/h_{ex} converges strongly in $W^{1,p}(\Omega)$ to the solution of*

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu \text{ in } \Omega \\ h_\mu = 1 \text{ on } \partial\Omega. \end{cases}$$

Moreover the symmetric 2-tensor T_μ with coefficients

$$T_{ij} = -\partial_i h_\mu \partial_j h_\mu + \frac{1}{2}\left(|\nabla h_\mu|^2 + {h_\mu}^2\right)\delta_{ij}$$

is divergence-free in finite part.

3. $h_{ex} = o(n_\varepsilon)$*, and then for some* $p \in (1,2)$*,* $\mu_\varepsilon/n_\varepsilon$ *converges in* $W^{-1,p}(\Omega)$ *to a measure* μ *and* $h_\varepsilon/n_\varepsilon$ *converges strongly in* $W^{1,p}(\Omega)$ *to the solution of*

$$\begin{cases} -\Delta U_\mu + U_\mu = \mu \text{ in } \Omega \\ U_\mu = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover, the symmetric 2-tensor T_μ *with coefficients*

$$T_{ij} = -\partial_i U_\mu \partial_j U_\mu + \frac{1}{2}\left(|\nabla U_\mu|^2 + {U_\mu}^2\right)\delta_{ij}$$

is divergence-free in finite part.

We will give in Chapter 13, Theorem 13.1, a version of this theorem applicable to general boundary conditions, which allows one to localize this result.

When $T_\mu \in L^1$, the fact that T_μ is divergence-free in finite part means that $\operatorname{div} T_\mu = 0$ in the sense of distributions, i.e., that $\partial_1 T_{i1} + \partial_2 T_{i2} = 0$ for $i = 1, 2$. If T_μ is not integrable, which is the case if μ is a Dirac mass for instance, the precise definition is a bit more complicated. If (but only if) h_μ is smooth enough, this is equivalent to the fact that

$$\mu \nabla h_\mu = 0.$$

This is the desired necessary condition on the limiting vorticity measure: it is a stationarity condition on the vortices, saying that on the support of μ, the limiting average current ∇h_μ must be 0 (see one possibility of density μ sketched in Fig. 1.9). If on the other hand, the number of vortices is small compared to the applied field (case 1), then (1.10) shows that vortices can only concentrate near the critical points of h_0 (defined in (1.6)), i.e., a finite set of points, see Fig. 1.10.

The analysis we develop in Chapter 13 allows us to treat the case of Ginzburg–Landau without magnetic field as well and find an analogue of this theorem.

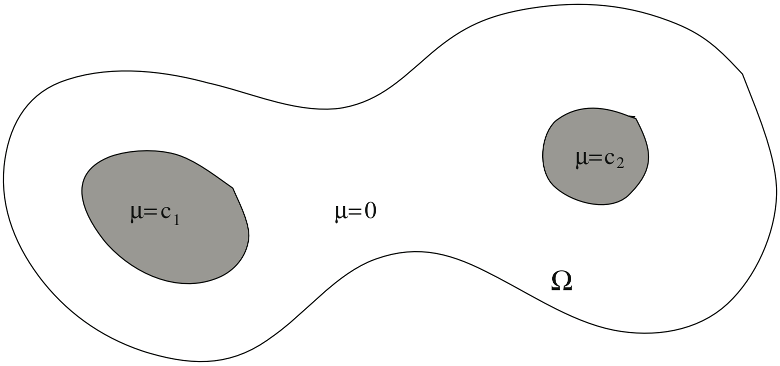

Figure 1.9: A possible limiting density.

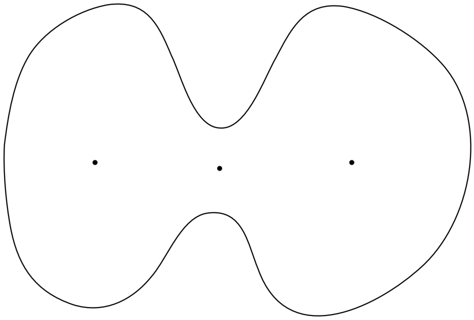

Figure 1.10: Critical points of h_0.

We have not cited much of the large mathematical literature here, but we refer to the end of the book, where we included a (necessarily incomplete) "guide to the literature" which schematically describes the results that have been obtained in the various branches of studies on Ginzburg–Landau problems. The book also ends with a list of open problems.

Let us sum up with a chart of results (and questions):

Regime	Type of Solutions	Limiting Measure	Reference Result
$h_{\rm ex} = \lambda \lvert \log \varepsilon \rvert$, $n \sim C h_{\rm ex}$.	minimizers	minimizers of E_λ	Theorem 7.2
$h_{\rm ex} \gg \lvert \log \varepsilon \rvert$, $2\pi n \sim h_{\rm ex}$.	minimizers	uniform measure dx	Theorem 8.1
$h_{\rm ex} \gg 1$, $n \sim C h_{\rm ex}$.	critical points	stationary points of E_λ or $\operatorname{div} T_\mu = 0$	Theorem 13.1
$1 \ll n \ll h_{\rm ex}$.	minimizers	minimizers of I	Theorem 9.2
$n \ll h_{\rm ex}$.	critical points	$\mu \nabla h_0 = 0$, what after blow-up?	Theorem 13.1
$n = O(1)$.	(local) minimizers	minimizers of w_n or $R_{n,h_{\rm ex}}$	Theorem 11.1
$h_{\rm ex} = O(1)$, $n = O(1)$.	critical points	"vanishing gradient property"	Theorem 13.1
$n \gg h_{\rm ex}$.	critical points	$\mu = 0$ where regular	Theorem 13.1

Chapter 2

Physical Presentation of the Model — Critical Fields

We begin by describing how the expression (1.1) for the Ginzburg–Landau functional is deduced from the expression (2.1) below, more commonly found in the physics literature. We will also give a nonrigorous introduction to critical fields in $\mathbb{R}^2$, in the spirit of Abrikosov, and draw a corresponding phase diagram in the $(\varepsilon, h_{\text{ex}})$ plane, i.e., qualitatively describe minimizers of the Ginzburg–Landau energy for different values of ε and h_{ex}, emphasizing the role of the vortices. Three areas of the parameter plane will be found: the normal, superconducting and mixed states, separated by what are usually called *critical lines.*

This chapter is meant to make the reader more familiar with the problems dealt with in the later chapters, and can either be read independently, or skipped by the reader wishing to get more quickly to the point.

2.1 The Ginzburg–Landau Model

Let us start with some notation. Given two complex numbers z, w, we let $(z, w) = \frac{1}{2}(\bar{z}w + z\bar{w})$, which is the inner product of z and w seen as vectors in $\mathbb{R}^2$. Partial derivatives are written $\partial_1 u$, $\partial_2 u, \ldots$. We will also write ∂_k^A for $\partial_k - iA_k$.

Consider a domain Ω in $\mathbb{R}^3$. In the Ginzburg–Landau model, the energy of a superconductor occupying Ω in the presence of a constant applied field H_e, when the exterior region is insulating, is

$$G(u,A) = G_0 + \int_{\mathbb{R}^3} \frac{|\operatorname{curl} A - H_e|^2}{8\pi} + \int_{\Omega} \frac{1}{2m^*} \left| \left(\hbar\nabla - \frac{ie^*}{c} A \right) u \right|^2 + \alpha |u|^2 + \beta |u|^4. \quad (2.1)$$

In this expression, $u : \Omega \to \mathbb{C}$ is the *order parameter* whose physical meaning is that of a "wave function" for superconducting electron pairs and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the electromagnetic vector potential, whose curl is the induced magnetic field. Besides the physical constants $\hbar$ and c, additional constants m^* and e^* are present (see [192] for an explanation of these constants) as well as two quantities α and β that depend on the temperature T and on the superconducting material. Near the so-called critical temperature T_c, it is assumed that β is a positive constant and α is proportional to $T - T_c$ and has the same sign. The quantity G_0 represents the energy of the normal state and, most important to us, does not depend on u or A.

2.1.1 Nondimensionalizing

The following changes of variable (assuming $0 \in \Omega$) make (2.1) more pleasant:

$$\tilde{u}(x) = \sqrt{\frac{\beta}{|\alpha|}} u(\lambda x), \qquad \tilde{A}(x) = \frac{e^*}{\hbar c} \lambda A(\lambda x), \qquad \widetilde{H_e} = \frac{e^*}{\hbar c} H_e, \quad (2.2)$$

where λ is the *penetration depth* defined by

$$\lambda = \sqrt{\frac{\beta m^* c}{4\pi |\alpha| e^{*2}}}.$$

We also introduce the *coherence length*

$$\xi = \hbar \sqrt{m^* |\alpha|}.$$

The energy then takes the form

$$\widetilde{G_0} + C \left[\frac{1}{2} \int_{\mathbb{R}^3} \left| \operatorname{curl} \tilde{A} - \widetilde{H_e} \right|^2 + \frac{1}{2} \int_{\tilde{\Omega}} \left| \left(\nabla - i\tilde{A} \right) \tilde{u} \right|^2 + \frac{1}{2\varepsilon^2} \left(1 \pm |\tilde{u}|^2 \right)^2 \right]$$

over a rescaled domain $\tilde{\Omega} = \Omega/\lambda$, where $\kappa = 1/\varepsilon = \lambda/\xi$ is the Ginzburg–Landau parameter which depends on the material and varies little with temperature, and $\widetilde{G}_0$ is independent of $\tilde{u}$ and $\tilde{A}$. The sign in $\left(1 \pm |u|^2\right)^2$ is the sign of the parameter α, i.e., is $+1$ if $T > T_c$ and -1 if $T < T_c$. In the first case, the functional is strictly convex hence clearly has a unique critical point, namely $\tilde{u} \equiv 0$ and $\tilde{A}$ such that $\operatorname{curl} \tilde{A} \equiv \widetilde{H_e}$. We are interested in the second case, where the phenomenon of superconductivity appears.

From now on we take $T < T_c$, assume the rescaling (2.2) and write u, A, H_e instead of $\tilde{u}, \tilde{A}, \widetilde{H_e}$ for the rescaled quantities. In this scaling the unit length is the penetration depth. The object of our study is therefore

$$\mathrm{GL}(u,A) = \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A - H_e|^2 + \frac{1}{2}\int_{\Omega} |(\nabla - iA)\,u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2.$$

Here $(\nabla - iA)\,u$ is the complex vector $(\partial_1^A u, \partial_2^A u, \partial_3^A u)$, where $\partial_k^A u = \partial_k u - iA_k u$.

The local state of the material at a point x is described by $u(x)$, the so-called order parameter. In this nondimensional form, $|u|^2(x)$ is the local density of superconducting electrons (the "Cooper pairs" of electrons). As in Landau theories, the state of the material is described through "phases", $|u| \simeq 1$ corresponds to the *superconducting phase* and $|u| \simeq 0$ to the *normal phase.*

2.1.2 Dimension Reduction

Since the full 3D model is quite complex, we wish to reduce to 2 dimensions. A natural special case is that of the domain being an infinite cylinder in $\mathbb{R}^3$ and H_e parallel to the axis (like an infinitely long insulated wire). Assuming translational invariance of (u, A) and invariance with respect to reflections across a plane perpendicular to the axis, we have, taking the cylinder's axis as the third coordinate axis,

$$H_e = h_{\mathrm{ex}}(0,0,1),\ u(x,y,z) = u(x,y),\ A(x,y,z) = (A_1(x,y), A_2(x,y), 0).$$

Then, the Ginzburg–Landau energy of (u, A) per unit length is

$$\mathrm{GL}(u,A) = \frac{1}{2}\int_{\mathbb{R}^2} |\operatorname{curl} A - h_{\mathrm{ex}}|^2 + \frac{1}{2}\int_{\Omega} |(\nabla - iA)\,u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^2$ is the cross section of the cylinder, $h_{\mathrm{ex}} \geq 0$ is the intensity of the applied field, and $h := \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1$ is the induced magnetic field. Our main goal will be to study the minimizers and critical points of this functional, i.e., the solutions of the associated Euler–Lagrange equations, derived below in Proposition 3.6:

$$\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \\ -\nabla^{\perp} h = (iu, \nabla_A u) & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{in } \mathbb{R}^2 \setminus \Omega \\ \nu \cdot \nabla_A u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

for different values of $\varepsilon, h_{\mathrm{ex}}$.

2.1.3 Gauge Invariance

The Ginzburg–Landau functional (1.1), hence the system (2.4), is invariant under the so-called gauge transformations

$$u \to u e^{if}, \quad A \to A + \nabla f,$$

where f is any smooth real-valued function. What is more, configurations which are deduced from one another by a gauge transformation describe the same physical state, hence the physical quantities associated to a configuration (u, A) are invariant under these transformations. It is quite clear that $h = \operatorname{curl} A$ and $|u|$ are gauge invariant. It is also the case for the superconducting current

$$j = (iu, \nabla_A u) = \frac{i}{2}\left(u\overline{\nabla_A u} - \bar{u}\nabla_A u\right),$$

(or the vector with components $(iu, \partial_1 u - iA_1 u)$ and $(iu, \partial_2 u - iA_2 u)$). It is not difficult to check that if $|u|$ does not vanish, then $|u|$, h and j determine (u, A) up to a gauge transformation (in a simply connected domain). If u vanishes or in nonsimply connected domains, this is not completely the case, since the missing information is the topological degree of u.

2.2 Notation

For any smooth bounded domain in $\mathbb{R}^2$ and any $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ we let

$$F_\varepsilon(u, A, \Omega) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + h^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2, \tag{2.5}$$

$$G_\varepsilon(u, A, \Omega) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + (h - h_{\text{ex}})^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2, \tag{2.6}$$

where $h_{\text{ex}} > 0$ is the intensity of the applied magnetic field, and

$$\nabla_A u = \nabla u - iAu, \quad h = \operatorname{curl} A := \partial_1 A_2 - \partial_2 A_1.$$

When there is no ambiguity, we denote $F_\varepsilon(u, A) = F_\varepsilon(u, A, \Omega)$ and $G_\varepsilon(u, A) = G_\varepsilon(u, A, \Omega)$. Note that in the following chapter we will see that the minimization of GL reduces to the minimization of G_ε when Ω is simply connected.

Even though G_ε depends on the parameter h_{ex} as well as on $\varepsilon(= \kappa^{-1})$, we do not reflect this in our notation because our main interest is in the asymptotics of the functional as ε tends to zero. In this limit, h_{ex} will be a function of ε and not an independent parameter. When it bears no importance, the subscript ε itself will be dropped.

We denote by $|\Omega|$ the two-dimensional Lebesgue measure of any measurable set Ω.

2.3 Constant States in $\mathbb{R}^2$

In the rest of the chapter, unless stated otherwise, we suppose for simplicity that the superconductor occupies the domain $\Omega = \mathbb{R}^2$, which corresponds to an infinitely large sample. We do not aim at mathematical rigor, but rather at explaining by formal calculations the notions of "critical fields" and "phase transitions".

If $\Omega = \mathbb{R}^2$, boundary conditions should be ignored and the system (2.4) reduces to the first two equations. We distinguish two solutions.

The superconducting solution for which $|u| \equiv 1$ and $h = \operatorname{curl} A \equiv 0$. It has infinite energy if $h_{\text{ex}} > 0$, but its energy density is ${h_{\text{ex}}}^2/2$. Note that all configurations satisfying the above are equivalent modulo gauge transformations, which is why we speak of *one* solution.

The normal solution. If $u \equiv 0$ and $h = \operatorname{curl} A$ is also a constant, then (u, A) is a solution. Its energy density is $\frac{1}{2}(h - h_{\text{ex}})^2 + \frac{1}{4\varepsilon^2}$, thus among these solutions, the least energetic is the one for which $h = h_{\text{ex}}$. If $u \equiv 0$ and A is such that $\operatorname{curl} A \equiv h_{\text{ex}}$, then (u, A) is called the normal solution.

Therefore we find a first critical line

$$\boxed{H_c(\varepsilon) = \frac{1}{\varepsilon\sqrt{2}}}, \tag{2.7}$$

meaning that if for a given value of ε we have $h_{\text{ex}} < H_c(\varepsilon)$, then the superconducting solution is more favorable than the normal one, whereas if $h_{\text{ex}} > H_c(\varepsilon)$, it is the reverse.

2.4 Periodic Solutions

The normal solution satisfies $u = 0$ everywhere. Abrikosov (see [1]) investigated the existence of solutions near the normal solution (in mathematical language, bifurcated solutions). He first showed that given ε, the largest value of h_{ex} for which the linearized equations about the normal solution admit solutions is the critical value

$$\boxed{H_{c_2}(\varepsilon) = \frac{1}{\varepsilon^2}}.$$

Moreover, from formal calculations which amount to a bifurcation analysis he argued that when $\varepsilon < \sqrt{2}$ these solutions give rise to a branch of solutions of the nonlinear equations when h_{ex} decreases below H_{c_2}, and that on these branches, the Ginzburg–Landau energy was lower than that of the normal solution. Recently, Dutour [89] rigorously showed the existence of these branches.

The Abrikosov solutions are *periodic*, or rather are such that the gauge-invariant quantities, such as $|u|$ and $h = \operatorname{curl} A$ are periodic. The zeroes of u form a lattice and around each zero u has a nonzero *degree* (or winding number). That is, writing $u = |u|e^{i\varphi}$, and working in polar coordinates (r, θ) centered at a zero of u, if $r > 0$ is small enough, the integer

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{\partial\varphi}{\partial\theta}(r, \theta)\, d\theta = \frac{1}{2\pi}\int_{\partial B(0,r)} \frac{1}{|u|^2}(iu, \partial_\tau u)$$

is not zero. The points where u vanishes are called *vortices* and the integer above, the *degree* of the vortex. At a vortex the induced magnetic field $h = \operatorname{curl} A$ has a local maximum.

There are many such solutions corresponding to different lattices in $\mathbb{R}^2$. Abrikosov [1] guessed that the one corresponding to a square lattice was the most favorable energetically, based on the fact that its expression as a power series was the simplest, but later numeric computations [127] showed that the hexagonal lattice was slightly better. We will see below that Abrikosov accurately predicted the hexagonal lattice near H_{c_1}, based on different considerations.

Remark 2.1. Note that when writing $u = |u|e^{i\varphi}$, the phase φ is not gauge-invariant, however the degree of a vortex is.

2.5 Vortex Solutions

Assume now $\varepsilon < \sqrt{2}$. If h_{ex} is large, the normal solution is more favorable than the superconducting or Abrikosov solutions. Then, lowering h_{ex} below H_{c_2}, the Abrikosov solutions become less energetic and the minimizer of the Ginzburg–Landau energy is supposedly one of them. The question is then to compute the critical value of h_{ex} below which the superconducting solution becomes in turn more favorable than the Abrikosov solutions. There is no reason why this value should be given by (2.7), which was computed by comparing the normal and superconducting solutions. We call the new value H_{c_1}, it should be smaller than H_c.

To simplify matters we will not compare the superconducting solution to an Abrikosov type solution, but rather to a single vortex solution, or rather approximate solution. This replacement of a doubly periodic configuration with a rotationally symmetric one may seem a bit strange, but we will justify it at the end of this chapter. The price to pay for all these approximations and the ones to come is that the computations will yield results valid only if ε is small, the so-called high-κ limit (or London limit).

2.5.1 Approximate Vortex

Our approximate solution will have — except for the constant states — the maximal symmetry allowed by the equations, i.e., rotational symmetry. We will look for (u, A) in the form

$$u(r,\theta) = f(r)e^{i\theta}, \quad A(r,\theta) = g(r)(-\sin\theta, \cos\theta). \tag{2.8}$$

Remark 2.2. True radial solutions in $\mathbb{R}^2$ of the Ginzburg–Landau equations of degree n, of the form

$$u_n(r,\theta) = f_n(r)e^{in\theta}, \quad A_n(r,\theta) = g_n(r)(-\sin\theta, \cos\theta)$$

have been shown to exist by Plohr [151, 152], and Berger and Chen [35]. Their linear stability was investigated by Gustafson and Sigal in [106] who proved that, as conjectured by Jaffe and Taubes in [112], they are stable if $n = \pm 1$, and if $|n| \geq 2$ they are stable if $\varepsilon > \sqrt{2}$ and unstable if $\varepsilon < \sqrt{2}$.

Next, we argue that if ε is small, then for $G_\varepsilon(u, A)$ to be as small as possible, $|u|$ should be close to 1 except on a small set. Moreover, scaling arguments suggest that the area of this set should be of the order of ε^2. For this reason we let

$$f(r) = \begin{cases} \frac{r}{\varepsilon} & \text{if } r < \varepsilon \\ 1 & \text{otherwise.} \end{cases} \tag{2.9}$$

We now need to define g in a reasonable way. Since the definition of u was rather arbitrary, or so it may seem, we will try to do a better job with A. The best would be of course to solve the Ginzburg–Landau equation for A, i.e.,

$$-\nabla^\perp h = (iu, \nabla_A u),$$

where $h = \operatorname{curl} A$. If we write $u = \rho e^{i\varphi}$, — we will use the ansatz (2.8) in a while — then

$$\nabla u = \nabla\rho e^{i\varphi} + i\rho\nabla\varphi e^{i\varphi} - iA\rho e^{i\varphi}$$

therefore $(iu, \nabla_A u) = \rho^2(\nabla\varphi - A)$.

Thus when $\rho = 1$, the second Ginzburg–Landau equation is $-\nabla^\perp h = \nabla\varphi - A$, and taking the curl yields

$$-\Delta h + h = 0. \tag{2.10}$$

When ρ varies, the equation for h is more complicated, but since this happens in a very small area, we will account for it in a simplified way. We compute

$$-\int_{B(0,\varepsilon)} \Delta h = -\int_{\partial B(0,\varepsilon)} \nu \cdot \nabla h = \int_{\partial B(0,\varepsilon)} \tau \cdot \nabla^\perp h.$$

Assuming the second Ginzburg–Landau equation is satisfied together with (2.8) and (2.9), we find

$$-\int_{B(0,\varepsilon)} \Delta h = \int_{\partial B(0,\varepsilon)} \tau \cdot (\nabla\varphi - A) = \int_{\partial B(0,\varepsilon)} \tau \cdot \nabla\theta - \int_{B(0,\varepsilon)} h.$$

Therefore

$$\int_{B(0,\varepsilon)} -\Delta h + h = 2\pi. \tag{2.11}$$

In view of (2.10)–(2.11), which we recall are consequences of our ansatz, together with the second Ginzburg–Landau equation, we *define* h to be the positive solution to

$$-\Delta h + h = 2\pi\delta, \tag{2.12}$$

where δ is the Dirac mass at 0. The solution is a radial function in $\mathbb{R}^2$. We deduce A in the form (2.8) from the relation $h = \operatorname{curl} A$ by integrating it over the ball $B(0,r)$. This yields

$$\int_{\partial B(0,r)} A \cdot \tau = \int_{B(0,r)} h$$

and then, together with (2.12),

$$g(r) = \frac{1}{r} + h'(r). \tag{2.13}$$

2.5.2 The Energy of the Approximate Vortex

We compute the energy of the configuration (u, A) defined by (2.8), (2.9), (2.12), (2.13). The energy in $\mathbb{R}^2$ is infinite, but we are really interested in the *difference* between the energy of (u, A) and that of the superconducting solution. Thus, writing B_r for $B(0,r)$, we let

$$\Delta(R) = \mathrm{GL}(u, A, B_R) - \mathrm{GL}(1, 0, B_R) = \mathrm{GL}(u, A, B_R) - \frac{1}{2}|B_R|{h_{\mathrm{ex}}}^2,$$

and try to compute the limit of this quantity as R tends to $+\infty$. As in (2.6), we have used the notation $G_\varepsilon(u, A, B_R)$ for the Ginzburg–Landau energy density integrated over the ball B_R. We split $\Delta(R)$ by writing $\Delta(R) = \alpha + \beta(R)$ for any $R > \varepsilon$, where

$$\alpha = \mathrm{GL}(u, A, B_\varepsilon) - \mathrm{GL}(1, 0, B_\varepsilon),$$
$$\beta(R) = G_\varepsilon(u, A, B_R \setminus B_\varepsilon) - \mathrm{GL}(1, 0, B_R \setminus B_\varepsilon).$$

To evaluate α and $\beta(R)$ we will need the following (see [192]):

Lemma 2.1. *Let h be the positive solution to $-\Delta h + h = 2\pi\delta$. Then $h(r) = |\log r| + C + o(1)$ as $r \to 0$ and the corresponding behavior for the derivative also holds, i.e., $h'(r) = -1/r + o(1)$ as $r \to 0$. Moreover $h(r), h'(r) = O(e^{-r})$ as $r \to +\infty$.*

Now we can prove:

Lemma 2.2. *Assuming $h_{ex} \le 1/\varepsilon^2$, there exists a constant C independent of $\varepsilon < 1$ such that $|\alpha| < C$.*

Proof. We let C denote a generic constant independent of $\varepsilon < 1$. From (2.8), (2.9) we have $|\nabla u| < C/\varepsilon$ in $\mathbb{R}^2$. From (2.8), (2.13) and Lemma 2.1, we find $|A| < C$ in B_1 and $\|h\|_{L^q(B_1)} < C$ for any $q \ge 1$. Therefore, in B_ε,

$$|\nabla_A u|^2 \le \frac{C}{\varepsilon^2}, \quad \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 \le \frac{C}{\varepsilon^2}$$

and since $-2h_{\mathrm{ex}} \le (h - h_{\mathrm{ex}})^2 - {h_{\mathrm{ex}}}^2 \le h^2$,

$$\int_{B_\varepsilon} \left|(h - h_{\mathrm{ex}})^2 - {h_{\mathrm{ex}}}^2\right| \le C.$$

It follows that

$$|\alpha| = |\mathrm{GL}(u, A, B_\varepsilon) - \mathrm{GL}(1, 0, B_\varepsilon)| = \left|\mathrm{GL}(u, A, B_\varepsilon) - \frac{{h_{\mathrm{ex}}}^2}{2}|B_\varepsilon|\right| \le C.$$

□

Concerning $\beta(R)$ we have:

Lemma 2.3. *Let $\beta(h_{ex}, \varepsilon) = \lim_{R \to +\infty} \beta(R)$. Then*

$$\beta(h_{ex}, \varepsilon) = \pi\left(|\log \varepsilon| - 2h_{ex}\right)(1 + o(1)) + O(1),$$

where $o(1)$ and $O(1)$ are meant as $\varepsilon \to 0$ and are independent of h_{ex}.

Proof. In $\mathbb{R}^2 \setminus B_\varepsilon$ we have $|u| = 1$. Therefore as noted above, the second Ginzburg–Landau equation becomes $-\nabla^\perp h = \nabla\varphi - A$, thus

$$\mathrm{GL}(u, A, B_R \setminus B_\varepsilon) = \frac{1}{2} \int\limits_{B_R \setminus B_\varepsilon} |\nabla h|^2 + (h - h_{\mathrm{ex}})^2.$$

Therefore

$$\beta(R) = \frac{1}{2} \int\limits_{B_R \setminus B_\varepsilon} |\nabla h|^2 + h^2 - 2hh_{\mathrm{ex}}.$$

Integrating by parts and using (2.12) yields

$$\int\limits_{B_R \setminus B_\varepsilon} |\nabla h|^2 + h^2 = \int\limits_{\partial B_R} h\frac{\partial h}{\partial \nu} - \int\limits_{\partial B_\varepsilon} h\frac{\partial h}{\partial \nu} = 2\pi R h(R)h'(R) - 2\pi\varepsilon h(\varepsilon)h'(\varepsilon)$$

and using (2.12) again,

$$\int\limits_{B_R \setminus B_\varepsilon} h = \int\limits_{B_R \setminus B_\varepsilon} \Delta h = 2\pi R h'(R) - 2\pi\varepsilon h(\varepsilon).$$

Therefore $\beta(R) = \pi R h'(R)(h(R) - 2h_{\mathrm{ex}}) - \pi\varepsilon h'(\varepsilon)(h(\varepsilon) - 2h_{\mathrm{ex}})$. From Lemma 2.1, $h'(R)$ goes to zero exponentially fast and as $R \to +\infty$ and as $\varepsilon \to 0$ we have $h'(\varepsilon) = -1/\varepsilon + o(1)$, $h(\varepsilon) = |\log\varepsilon| + O(1)$. The lemma follows. □

2.5.3 The Critical Line H_{c_1}

In view of Lemmas 2.2, 2.3, We find that

$$\lim_{R \to +\infty} \Delta(R) = \pi \log \frac{1}{\varepsilon} - 2\pi h_{\mathrm{ex}} + C,$$

where C is bounded independently of ε. Clearly this result is meaningful only for small values of ε, but shows that in this case, as established by Abrikosov, there exists a critical value

$$\boxed{H_{c_1}(\varepsilon) \approx \tfrac{|\log\varepsilon|}{2}} \tag{2.14}$$

such that if h_{ex} is below $H_{c_1}(\varepsilon)$, the superconducting solution is energetically favorable compared to the approximate vortex whereas it is the opposite if $h_{\mathrm{ex}} > H_{c_1}(\varepsilon)$.

Several remarks can be made at this point. First, the equivalent for H_{c_1} as $\varepsilon \to 0$ that we computed is not very sensitive to the way we construct the approximate vortex. We see from Lemma 2.2 for instance that the contribution of B_ε is negligible when computing the value $|\log \varepsilon|/2$.

The second remark is that the approximate vortex is quite different from the Abrikosov solutions, should not there be other critical values of h_{ex} marking the transition from one vortex to two vortices and so on? The answer is positive in the case of a bounded domain; but in $\mathbb{R}^2$, although the approximate vortex allows one to compute the right critical value, the least energy configuration when h_{ex} crosses the line should look more like a vortex lattice similar to an Abrikosov solution. The reason for this is that if adding a vortex to the superconducting solution allows one to gain some energy, then adding many vortices allows one to gain more energy. The minimizer will then be a lattice of vortex solutions glued together. As h_{ex} decreases to H_{c_1} the density of the lattice will decrease to 0: vortices grow infinitely far from each other. For rigorous results on the analysis of periodic solutions to Ginzburg–Landau around the critical field H_{c_1}, see [28].

It is interesting to note that near H_{c_1}, Abrikosov guessed that the energy minimizers would exhibit vortices arranged in a hexagonal lattice, the one for which given the cell area of the lattice, the closest points are the farthest away possible. Indeed, putting vortices far apart makes their gluing together more efficient in terms of energy. This was inconsistent with his prediction of a square lattice near H_{c_2}, raising the question of the transition from square to hexagonal, but the hexagonal lattice finally proved better near H_{c_2} as well.

2.6 Phase Diagram

We may sum up the previous analysis in the diagram of Fig. 2.1, where we have plotted the critical lines in the plane (x, y), where $x = \kappa = 1/\varepsilon$ is the Ginzburg–Landau parameter and $y = h_{\text{ex}}/\kappa$. To the left of $\kappa = 1/\sqrt{2}$, the Abrikosov solutions do not exist and there is a single critical line separating the domains where the energy minimizer is respectively the normal and superconducting solution. When $\kappa > 1/\sqrt{2}$, the critical line H_c divides into two: the critical line $H_{c_2}(\kappa) = \kappa^2$ above which the normal solution is the minimizer, and the critical line $H_{c_1}(\kappa)$ which behaves for large κ as $\frac{1}{2}\log\kappa$ and below which the superconducting solution is the minimizer. In between these two lines we expect the minimizer to be an

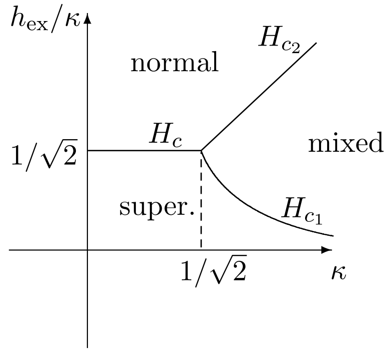

Figure 2.1: Phase diagram in $\mathbb{R}^2$.

Abrikosov type solution, i.e., a lattice of vortices. This state where the superconducting phase $|u| \sim 1$ and the normal phase $|u| \sim 0$ (under the form of vortices) coexist, is called in physics the "mixed state".

The separation at $\varepsilon = \sqrt{2}$ or $\kappa = \frac{1}{\sqrt{2}}$ (recall that $\varepsilon = 1/\kappa$ is a material constant) corresponds to the distinction between type-I ($\kappa < \frac{1}{\sqrt{2}}$) and type-II ($\kappa > \frac{1}{\sqrt{2}}$) superconducting materials, which have different qualitative behavior (as we just saw there is no mixed state in type-I superconductors). However, the value of this threshold of separation is really valid for infinite samples. At $\kappa = \frac{1}{\sqrt{2}}$, the Ginzburg–Landau equations become self-dual and decouple into two first order equations (see [112]).

2.6.1 Bounded Domains

In the case of bounded domains, which will be our focus, the situation is roughly similar, except for various boundary effects. In particular, there is a third critical field H_{c_3} larger than H_{c_2} at which the bifurcation from the normal solution happens through *surface superconductivity.* We refer to Chapter 14 for references on this. Another finite size effect is that, even though there still exists a pure superconducting solution, called the Meissner solution, it is no longer a constant. There still exist vortex solutions, but these of course can no longer be truly periodic nor found explicitly. At the first critical field H_{c_1}, which is larger than the one found for the infinite domain, there exists a similar phase-transition from superconducting state to vortex state except that the vortices appear one by one, near the center. This will be described in detail in the book.

The rest of this book will be devoted, roughly speaking, to the study of minimizers and critical points of the Ginzburg–Landau functional in the range of parameters κ large and h_{ex} well below H_{c_2}, that is well below $1/\varepsilon^2$. As a byproduct we will, for instance, provide a rigorous derivation of (2.14) (or rather of its analogue for bounded domains — we will thus observe the influence of boundary) from the minimization of the Ginzburg–Landau functional.

Bibliographic notes on Chapter 2: The material presented in this chapter is fairly standard in the physics literature. The reader may refer to the standard textbooks on superconductivity, such as Tinkham [192], Saint-James–Sarma–Thomas [164], and DeGennes [80]. One may also see the lectures of Rubinstein [158].

Chapter 3

First Properties of Solutions to the Ginzburg–Landau Equations

In this chapter, we start to investigate the mathematical aspects of the Ginzburg–Landau energy and equations. Whereas the material in the first three sections is relatively easy or standard (existence of minimizers, regularity of solutions, apriori estimates ...) and used throughout the later chapters, the material of the last two sections is more advanced, contains several results stated without proofs, and is only used in Chapter 5 and then Chapters 10 to 12. However, we feel that the material is important enough, like the uniqueness result of P. Mironescu (Theorem 3.2), or basic enough to deserve to be stated early on.

Here and in the rest of the book, $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω; $H_0^1(\Omega)$ denotes the closure of smooth functions with compact support in Ω in the H^1 norm $\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$. Similarly $W_0^{1,p}(\Omega)$ denotes the closure of smooth functions with compact support in Ω in the $W^{1,p}$ norm while $W^{-1,p}$ denotes the dual of $W_0^{1,q}$, where $1/p + 1/q = 1$, and H^{-1} the dual of H_0^1.

3.1 Minimizing the Ginzburg–Landau Energy

From now on, Ω is a smooth bounded simply connected domain in $\mathbb{R}^2$.

3.1.1 Coulomb Gauge

Definition 3.1. [Gauge equivalence] For any $H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ function f, any $u \in H^1(\Omega, \mathbb{C})$ and any $A \in H^1(\mathbb{R}^2, \mathbb{R}^2)$ we define

$$v = ue^{if}, \quad B = A + \nabla f$$

and we say the configuration (v, B) is *gauge-equivalent* to (u, A). The transformation from (u, A) to (v, B) is called a *gauge transformation.* If A is only defined in Ω, then we require f only to be defined in Ω and to be in $H^2(\Omega)$.

Proposition 3.1. *If (v, B) and (u, A) are gauge-equivalent (in $\mathbb{R}^2$), then $GL(v, B) = GL(u, A)$. If they are defined and gauge-equivalent in Ω, then $G_\varepsilon(v, B) = G_\varepsilon(u, A)$.*

Proof. If $v = ue^{if}$ and $B = A + \nabla f$ for some real-valued function f, then $\operatorname{curl} B = \operatorname{curl} A$, $|v| = |u|$ and

$$\nabla v = (\nabla u + iu\nabla f)\, e^{if}, \quad iBv = (iAu + iu\nabla f)\, e^{if}$$

hence $(\nabla - iB)\, v = e^{if}\, (\nabla - iA)\, u$. Replacing this in (2.3) and (2.6) proves the proposition. □

Remark 3.1. As stated before, essential gauge invariant quantities are $|u|$, h, and the superconducting current $j = (iu, \nabla_A u)$. It is an exercise to check that if (u, A) and (v, B) are such that $|u| = |v| > 0$, $(iu, \nabla_A u) = (iv, \nabla_B v)$ in a simply connected domain Ω and $\operatorname{curl} A = \operatorname{curl} B$ in $\mathbb{R}^2$, then they are gauge-equivalent.

This invariance of the energy by a large group of transformation (all smooth real-valued functions) poses a problem for the minimization of GL. Indeed if $\{(u_n, A_n)\}_n$ is a minimizing sequence, then for any sequence of functions $\{f_n\}_n$, $\{(u_n e^{if_n}, A_n + \nabla f_n)\}_n$ is also minimizing, however wild the functions f_n may be. Thus no good bounds on $\{(u_n, A_n)\}_n$ can be deduced from the fact that $\mathrm{GL}(u_n, A_n)$ is bounded independently of n. The use of a particular gauge, namely the Coulomb gauge, solves this problem.

Definition 3.2. [Coulomb gauge] Let Ω be a smooth bounded domain in $\mathbb{R}^2$. We say $A : \Omega \to \mathbb{R}^2$ satisfies the Coulomb gauge condition in Ω if

$$\begin{cases} \operatorname{div} A = 0 \text{ in } \Omega \\ A \cdot \nu = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where ν is the outward pointing unit normal to $\partial\Omega$.

We have:

Proposition 3.2. *For any smooth bounded domain $\Omega \subset \mathbb{R}^2$ and for any $A \in H^1(\Omega, \mathbb{R}^2)$, there exists a gauge transformation $f \in H^2(\Omega)$ such that $B = A + \nabla f$ satisfies the Coulomb gauge condition in Ω.*

Proof. Let f solve

$$\begin{cases} \Delta f = -\operatorname{div} A & \text{in } \Omega \\ \partial_\nu f = -A \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

This is possible since $\int_\Omega \operatorname{div} A = \int_{\partial\Omega} A \cdot \nu$ and the solution is unique modulo a constant. Then $A + \nabla f$ satisfies the desired conditions. $\square$

The following estimate is crucial for the minimization of (2.3):

Proposition 3.3. *Let Ω be a smooth, bounded, simply connected domain in $\mathbb{R}^2$. There exists a constant $C > 0$ such that if $A : \Omega \to \mathbb{R}^2$ satisfies the Coulomb gauge condition, then*

$$\|A\|^2_{H^1(\Omega,\mathbb{R}^2)} \le C \|\operatorname{curl} A\|^2_{L^2(\Omega)}$$

and

$$\|A\|^2_{H^2(\Omega,\mathbb{R}^2)} \le C \|\operatorname{curl} A\|^2_{H^1(\Omega)}.$$

Proof. Since Ω is simply connected and $\operatorname{div} A = 0$ in Ω, by Poincaré's lemma there exists a function f such that $A = (-\partial_2 f, \partial_1 f)$. Then $A \cdot \nu = 0$ on $\partial\Omega$ implies that f is constant on $\partial\Omega$ and, subtracting the constant, we may assume $f = 0$ on $\partial\Omega$. Moreover $\operatorname{curl} A = \Delta f$. Standard elliptic regularity then implies that $\|f\|^2_{H^2(\Omega)} \le C\|\operatorname{curl} A\|^2_{L^2(\Omega)}$ and $\|f\|^2_{H^3(\Omega)} \le C\|\operatorname{curl} A\|^2_{H^1(\Omega)}$, from which the result follows. $\square$

3.1.2 Restriction to Ω

The natural space for the minimization of (2.3) is

$$X = \{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \mid (\operatorname{curl} A - h_{\text{ex}}) \in L^2(\mathbb{R}^2)\}.$$

To avoid the technical difficulties of minimizing GL, as defined in (2.3), over X, we instead minimize G_ε as defined in (2.6) over the space

$$X_\Omega = \{(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)\}. \tag{3.2}$$

It is clear that if $(u, A) \in X$, then its restriction to Ω is in X_Ω and

$$G_\varepsilon(u, A) \leq \mathrm{GL}(u, A). \tag{3.3}$$

Conversely we have:

Lemma 3.1. *Let $(u, A) \in X_\Omega$. Then A can be extended to $\mathbb{R}^2$ in such a way that $G_\varepsilon(u, A) = GL(u, A)$.*

Proof. There exists $B \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{curl} B = \operatorname{curl} A$ in Ω and $\operatorname{curl} B = h_{\mathrm{ex}}$ outside Ω. For example, we can take $B = \nabla^\perp \phi$ where ϕ solves $-\Delta\phi = g$ with $g(x) = \operatorname{curl} A(x)$ if $x \in \Omega$ and $g(x) = h_{\mathrm{ex}}$ if not. Then, since Ω is simply connected, there exists a function $f \in H^2(\Omega)$ such that $B = A + \nabla f$ in Ω. It follows that $G_\varepsilon(u, A) = G_\varepsilon(ue^{if}, B)$ and since $\operatorname{curl} B = h_{\mathrm{ex}}$ outside Ω, we find $G_\varepsilon(u, A) = \mathrm{GL}(ue^{if}, B)$. By extending f to $\mathbb{R}^2$ in an arbitrary way to a function $f \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ and gauge transforming (ue^{if}, B) by $-f$, the lemma is proved. □

This lemma together with (3.3) proves:

Proposition 3.4. *The minimum of GL over X is equal to the minimum of G_ε over X_Ω. Moreover, minimizers of GL restrict to minimizers of G_ε and conversely, minimizers of G_ε can be extended to minimizers of GL.*

We prove below that a minimizer of G_ε, hence a minimizer of GL, exists.

3.1.3 Minimization of GL

Proposition 3.5. *The minimum of GL over X is achieved.*

Proof. From Proposition 3.4 it suffices to prove that the minimum of G_ε over X_Ω is achieved. Let $\{(u_n, A_n)\}_n$ be a minimizing sequence for G_ε. We may assume by density that the terms of the sequence are smooth. Also, using Proposition 3.2, we may assume A_n satisfies the Coulomb gauge condition in Ω for all n. Using the bound $G_\varepsilon(u_n, A_n) \leq C$, where C is independent of n, we find that $\|1 - |u_n|^2\|_{L^2(\Omega)}$, $\|(\nabla - iA_n)u_n\|_{L^2(\Omega)}$ and $\|\operatorname{curl} A_n - h_{\mathrm{ex}}\|_{L^2(\Omega)}$ are bounded independently of n. Therefore $\{\operatorname{curl} A_n\}_n$ is bounded in $L^2(\Omega)$ and thus, from Proposition 3.3, $\{A_n\}_n$ is bounded in $H^1(\Omega)$.

Note that $\nabla u_n = (\nabla - iA_n)u_n + iA_n u_n$. Since $\{A_n\}_n$ is bounded in $H^1(\Omega)$, it is bounded in every L^q for $q < \infty$ by Sobolev embedding.

Because $\{u_n\}_n$ is bounded in L^4 we find that $\{iA_n u_n\}_n$ is bounded in $L^{4-\eta}$ for any $\eta > 0$ and in particular in L^2. Thus $\{\nabla u_n\}_n$ is bounded in L^2 and $\{u_n\}_n$ is bounded in $H^1(\Omega)$.

We may then extract a subsequence such that $\{u_n\}_n$ and $\{A_n\}_n$ converge to some (u_0, A_0) weakly in $H^1(\Omega)$ and, by compact Sobolev embedding, strongly in every L^q for $q < \infty$. We now show that (u_0, A_0) is a minimizer of G_ε.

By strong L^4 convergence, $\liminf_n \|1-|u_n|^2\|^2_{L^2(\Omega)} = \|1-|u_0|^2\|_{L^2(\Omega)}$. Also, $\|\operatorname{curl} A - h_{\mathrm{ex}}\|^2_{L^2}$ is a convex function of A which is continuous in the H^1 norm, hence it is weakly lower semicontinuous in H^1. Therefore $\liminf_n \|\operatorname{curl} A - h_{\mathrm{ex}}\|^2_{L^2} \geq \|\operatorname{curl} A_0 - h_{\mathrm{ex}}\|^2_{L^2}$. It remains to check that

$$\liminf_n \|(\nabla - iA_n)u_n\|^2_{L^2} \geq \|(\nabla - iA_0)u_0\|^2_{L^2}. \tag{3.4}$$

Note that

$$|(\nabla - iA)u|^2 = |\nabla u|^2 - 2\,(A \cdot \nabla u, iu) + |A|^2|u|^2. \tag{3.5}$$

From the weak H^1 convergence of u_n to u_0, we first deduce

$$\liminf_n \int_\Omega |\nabla u_n|^2 \geq \int_\Omega |\nabla u_0|^2.$$

Secondly, combining the strong L^q convergence of u_n and A_n to the weak L^2 convergence of ∇u_n, we find

$$\lim_n \int_\Omega (A_n \cdot \nabla u_n, iu_n) = \int_\Omega (A_0 \cdot \nabla u_0, iu_0)$$

and thirdly, by strong L^q convergence of u_n and A_n again, that $\lim_n \int_\Omega |A_n|^2|u_n|^2 = \int_\Omega |A_0|^2|u_0|^2$. Combining the three and (3.5), we find (3.4). □

3.2 Euler–Lagrange Equations

Definition 3.3. [Critical point] We say that $(u, A) \in X$ is a critical point of GL if for every (v, B) smooth and compactly supported we have

$$\frac{d}{dt}\mathrm{GL}(u + tv, A + tB)_{|t=0} = 0.$$

Clearly, a minimizer of GL is a critical point.

Proposition 3.6. *If* $(u, A) \in X$ *is a critical point of GL then, letting* $h = \operatorname{curl} A$*, we have*

$$\begin{cases} -(\nabla_A)^2 u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = (iu, \nabla_A u) & \text{in } \Omega \\ h = h_{ex} & \text{in } \mathbb{R}^2 \setminus \Omega \\ \nu \cdot \nabla_A u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

If $(u, A) \in X_\Omega$ *is a critical point of* G_ε *then the same equations are satisfied, with* $h = h_{ex}$ *satisfied on* $\partial\Omega$ *instead of* $\mathbb{R}^2 \setminus \Omega$*.*

Note that the *covariant Laplacian* is defined by

$$(\nabla_A)^2 u = \partial_1^A(\partial_1^A u) + \partial_2^A(\partial_2^A u),$$

where we recall that $\partial_j^A u = \partial_j u - iA_j u$. The *covariant gradient* is

$$\nabla_A u = (\nabla - iA)u$$

and the *current* is the vector in $\mathbb{R}^2$ defined by

$$(iu, \nabla_A u) = \left((iu, \partial_1^A u), (iu, \partial_2^A u)\right),$$

where, for complex numbers $z = x + iy$, $w = x' + iy'$, we let $(z, w) = xx' + yy'$. Finally,

$$\nu \cdot \nabla_A u = \nu^1 \partial_1^A u + \nu^2 \partial_2^A u.$$

The derivation of (3.6) is made very close to, say, the derivation of the Laplace equation from the minimization of the Dirichlet energy, by using the following lemma, the proof of which is left to the reader.

Lemma 3.2. *For arbitrary complex-valued functions* u, v *and any* A*,*

$$\partial_k(u, v) = (\partial_k^A u, v) + (u, \partial_k^A v).$$

Proof of the Proposition. We have

$$\frac{d}{dt} \mathrm{GL}(u + tv, A + tB)_{|t=0} = \int_\Omega (\nabla_A u, \nabla_A v) + (\nabla_A u, -iBu) \\ - \int_\Omega \frac{(u, v)}{\varepsilon^2}(1 - |u|^2) + \int_{\mathbb{R}^2} (\operatorname{curl} A - h_{\mathrm{ex}}) \operatorname{curl} B,$$

where $(\nabla_A u, \nabla_A v) = (\partial_1^A u, \partial_1^A v) + (\partial_2^A u, \partial_2^A v)$. Using the lemma, we have

$$(\nabla_A u, \nabla_A v) = \sum_{k=1}^{2} \partial_k(\partial_k^A u, v) - ((\partial_k^A)^2 u, v) = \operatorname{div}(\nabla_A u, v) - ((\nabla_A)^2 u, v),$$

where $(\nabla_A u, v) = \big((\partial_1^A u, v), (\partial_2^A u, v)\big)$. Therefore, integrating by parts

$$\frac{d}{dt}\mathrm{GL}(u + tv, A + tB)_{|t=0} = \int_{\partial\Omega} (\nu \cdot \nabla_A u, v) + \int_{\Omega} - \big((\nabla_A)^2 u, v\big) - (iu, \nabla_A u) \cdot B - \frac{(u, v)}{\varepsilon^2}(1 - |u|^2) - \int_{\mathbb{R}^2} \nabla^\perp(\operatorname{curl} A - h_{\mathrm{ex}}) \cdot B.$$

Since this is true for any (v, B), we find $-\nabla^\perp(h - h_{\mathrm{ex}}) = (iu, \nabla_A u)$ and

$$-(\nabla_A)^2 u = \frac{u}{\varepsilon^2}(1 - |u|^2)$$

in Ω, while $\nabla^\perp(h - h_{\mathrm{ex}}) = 0$ outside Ω. Since h_{ex} is constant, h is constant outside Ω and this constant must be h_{ex} since the configuration has finite energy. The boundary conditions follow as well. □

A different but useful form of the system (3.6) is expressed by the fact that the divergence of the stress-energy tensor is zero.

Definition 3.4. [Stress-energy tensor] The stress-energy tensor associated to a configuration (u, A) for a given $\varepsilon > 0$ is the symmetric 2×2 tensor T with coefficients

$$T_{ij} = (\partial_i^A u, \partial_j^A u) - \frac{1}{2}\left(|\nabla_A u|^2 - h^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2\right)\delta_{ij},$$

where $h = \operatorname{curl} A$.

We have:

Proposition 3.7 (The stress-energy tensor is divergence-free). *Assume (u, A) is a critical point of G_ε. Then the stress-energy tensor T associated to (u, A) satisfies for $i = 1, 2$,*

$$\partial_1 T_{1i} + \partial_2 T_{2i} = 0$$

in Ω and we write in shorthand $\operatorname{div} T = 0$.

Proof. Using Lemma 3.2 we find

$$\left(\partial_1^A u, (\nabla_A)^2 u\right) = \frac{1}{2}\partial_1|\partial_1^A u|^2 + \partial_2(\partial_1^A u, \partial_2^A u) - (\partial_2^A\partial_1^A u, \partial_2^A u).$$

But

$$(\partial_2^A\partial_1^A u, \partial_2^A u) = \frac{1}{2}\partial_1|\partial_2^A u|^2 + \left((\partial_2^A\partial_1^A - \partial_1^A\partial_2^A)u, \partial_2^A u\right),$$

and it is a simple computation to check that $(\partial_2^A\partial_1^A - \partial_1^A\partial_2^A)u = iuh$, with $h = \operatorname{curl} A$. Therefore we find

$$\left(\partial_1^A u, (\nabla_A)^2 u\right) = \frac{1}{2}\partial_1\left(|\partial_1^A u|^2 - |\partial_2^A u|^2\right) + \partial_2(\partial_1^A u, \partial_2^A u) - (iu, \partial_2^A u)h.$$

From the second Ginzburg–Landau equation $-(iu, \partial_2^A u) = +\partial_1 h$ therefore

$$\left(\partial_1^A u, (\nabla_A)^2 u\right) = \frac{1}{2}\partial_1\left(|\partial_1^A u|^2 - |\partial_2^A u|^2\right) + \partial_2(\partial_1^A u, \partial_2^A u) + \frac{1}{2}\partial_1 h^2.$$

Now if we take the scalar product of the first Ginzburg–Landau equation with $\partial_1^A u$, we find

$$-\frac{1}{2}\partial_1\left(|\partial_1^A u|^2 - |\partial_2^A u|^2\right) - \partial_2(\partial_1^A u, \partial_2^A u) - \frac{1}{2}\partial_1 h^2 = \frac{1}{\varepsilon^2}\left(\partial_1^A u, u\right)\left(1 - |u|^2\right)^2.$$

Since $(\partial_1^A u, u) = (\partial_1 u, u) = \partial_1|u|^2/2$, we finally obtain that

$$-\frac{1}{2}\partial_1\left(|\partial_1^A u|^2 - |\partial_2^A u|^2\right) - \partial_2(\partial_1^A u, \partial_2^A u) - \frac{1}{2}\partial_1 h^2 + \frac{1}{4\varepsilon^2}\partial_1\left(1 - |u|^2\right)^2 = 0$$

which is exactly $\partial_1 T_{11} + \partial_2 T_{21} = 0$. The relation $\partial_1 T_{12} + \partial_2 T_{22} = 0$ is proved in the same way. □

3.3 Properties of Critical Points

Proposition 3.8 (Regularity). *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If (u, A) is a critical point of G_ε and if A satisfies the Coulomb gauge condition* (3.1)*, then u and A are smooth in Ω.*

Proof. Together with the Coulomb gauge condition, the Ginzburg–Landau equations (3.6) become

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) - 2i(A\cdot\nabla)u - |A|^2 u & \text{in } \Omega \\ -\Delta A = (iu, \nabla u - iAu) & \text{in } \Omega \\ \quad h = h_{\text{ex}} & \text{on } \partial\Omega \\ \nu\cdot\nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

The first equation is obtained by expanding $(\nabla_A)^2 u$. To obtain the second equation from (2.4), note that

$$-\nabla^\perp h = \left(\partial_2(\partial_1 A_2 - \partial_2 A_1), -\partial_1(\partial_1 A_2 - \partial_2 A_1)\right). \tag{3.8}$$

Differentiating $\partial_1 A_1 + \partial_2 A_2 = 0$ with respect to both variables we find $\partial_{12} A_2 = -\partial_{11} A_1$ and $\partial_{12} A_1 = -\partial_{22} A_2$. Replacing this in (3.8) yields $-\nabla^\perp h = -\Delta A$ and thus (3.7).

But (3.7) is a couple of elliptic equations for which we easily derive regularity by bootstrapping arguments. Since (u, A) are both in $H^1(\Omega)$, hence in every L^q, the right-hand side of the equations (3.7) are in L^p for any $p < 2$ and therefore (u, A) are both in $W^{2,p}$ by standard elliptic theory, and therefore in every $W^{1,q}$, etc. □

Boundary regularity can be recovered in a similar way, once it is checked that the boundary conditions above satisfy the so-called complementing condition (see [7]). To see why they do, assume for simplicity that Ω is the half space $\{(x,y)\in\mathbb{R}^2 \mid x > 0\}$. Then, writing $A = (A_1, A_2)$, the Coulomb gauge condition $A\cdot\nu = 0$ at the boundary becomes $A_1 = 0$ on $\partial\Omega$ and then $\operatorname{curl} A = h_{\text{ex}}$ becomes $\partial_\nu A_2 = h_{\text{ex}}$. Therefore we have a Dirichlet condition for A_1 and a Neumann condition for A_2. This is almost a proof that the complementing condition is satisfied. The rest of the proof of the boundary regularity consists in a bootstrapping argument as above.

The reference [85] discusses these issues, without however giving a complete proof of the boundary regularity.

Proposition 3.9. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If (u, A) is a critical point of G_ε, then $|u| \le 1$ in Ω.*

Proof. This is a consequence of the maximum principle. Taking the scalar product of the first equation in (3.7) with u we find

$$-(\Delta u, u) = \frac{1}{\varepsilon^2}|u|^2(1-|u|^2) - 2(i(A\cdot\nabla)u,u) - |A|^2|u|^2.$$

Therefore

$$\begin{aligned}\frac{1}{2}\Delta|u|^2 &= (\Delta u, u) + |\nabla u|^2 \\ &= \frac{-1}{\varepsilon^2}|u|^2(1-|u|^2) + 2(i(A\cdot\nabla)u,u) + |A|^2|u|^2 + |\nabla u|^2.\end{aligned}$$

Noting that

$$|\nabla_A u|^2 = |\nabla u|^2 + 2(i(A\cdot\nabla)u,u) + |A|^2|u|^2$$

we find

$$-\frac{1}{2}\Delta|u|^2 = \frac{1}{\varepsilon^2}|u|^2(1-|u|^2) - |\nabla_A u|^2. \tag{3.9}$$

Let us now consider x_0 a point of maximum of $|u|$ in $\overline{\Omega}$. Since u is smooth in view of Proposition 3.8, we can write that if $x_0 \in \Omega$, $\nabla|u|(x_0) = 0$ and $\Delta|u|(x_0) \le 0$, hence we deduce from (3.9) that $1 - |u|^2(x_0) \ge 0$ and thus $|u|(x_0) \le 1$. If on the other hand $x_0 \in \partial\Omega$, then $\frac{\partial|u|}{\partial\tau}(x_0) = 0$. Moreover, the Neumann boundary condition $\nu\cdot\nabla_A u = 0$ implies, taking the scalar product with u, that $\partial_\nu|u|(x_0) = 0$. Therefore, $\nabla|u|(x_0) = 0$ and we can argue similarly that $\Delta|u|(x_0) \le 0$, implying $|u|(x_0) \le 1$. We conclude in all cases that $\max|u| \le 1$. □

The following result follows directly.

Lemma 3.3. *If* (u, A) *is a solution to (3.6), then*

$$|j| = |\nabla h| \le |\nabla_A u|.$$

Proof. From the second Ginzburg–Landau equation we have pointwise $|\nabla h| \le |(iu, \nabla_A u)|$ and since $|u| \le 1$ from Proposition 3.9, we find $|\nabla h| \le |\nabla_A u|$. □

This will often be used combined with the result.

Lemma 3.4. *Assume u is defined and differentiable in a neighborhood of x and takes values in $\mathbb{C}$. If $u(x) \neq 0$, then u can be written in a neighborhood of x as $\rho e^{i\varphi}$, where ρ, φ are real valued and ρ is positive. Then in this neighborhood*

$$\nabla_A u = \rho i e^{i\varphi}(\nabla\varphi - A) + e^{i\varphi}\nabla\rho.$$

In particular

$$|\nabla_A u|^2 = \rho^2|\nabla\varphi - A|^2 + |\nabla\rho|^2,$$

and

$$j = \rho^2(\nabla\varphi - A).$$

Proposition 3.10. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. If (u, A) is a critical point of G_ε, then*

$$\|h - h_{ex}\|^2_{H^1(\Omega)} \leq 2G_\varepsilon(u, A)$$
$$\|h\|^2_{H^1(\Omega)} \leq 2F_\varepsilon(u, A)$$
$$\|A\|^2_{H^2(\Omega)} \leq CF_\varepsilon(u, A),$$

where we recall F_ε is defined in (2.5).

Proof. Using Lemma 3.3, squaring and integrating, adding $\int_\Omega (h - h_{\text{ex}})^2$ or $\int_\Omega h^2$ on both sides yields

$$\int_\Omega |\nabla h|^2 + |h - h_{\text{ex}}|^2 \leq 2G_\varepsilon(u, A)$$
$$\int_\Omega |\nabla h|^2 + h^2 \leq 2F_\varepsilon(u, A),$$

and the third assertion follows from Proposition 3.3. □

Let us mention here a property of the zeroes of energy-minimizers (for simply-connected domains):

Theorem 3.1 (Elliott–Matano–Tang Qi [92]**).** *Let (u, A) be a minimizer of G_ε. Then the set of zeroes of u consists only of isolated points.*

3.4 Solutions in the Plane

The parameter ε in the Ginzburg–Landau equations is the lengthscale on which the order parameter u varies. It is therefore interesting to study the blow-ups of solutions at this scale. It turns out that if one chooses to work in the Coulomb gauge, the blow-up limits as $\varepsilon \to 0$ satisfy $-\Delta u = u(1 - |u|^2)$ in $\mathbb{R}^2$. In this section, we list some properties of these limits before proving the convergence of blow-up sequences in the following section. We begin by collecting some facts about the topological degree.

3.4.1 Degree Theory

Degree of $\mathbb{S}^1$-valued maps

Assume Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, with its natural orientation. We let τ denote the unit tangent vector to $\partial\Omega$ compatible with this orientation.

Definition 3.5. If $u : \partial\Omega \to \mathbb{S}^1$ is a sufficiently regular map, the *degree* of u is defined by

$$\deg(u, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} (iu, \partial_\tau u)\, ds. \tag{3.10}$$

Assuming u to be smooth, it can be written locally as $u = \exp(i\varphi)$ for some smooth real-valued function φ (a "lifting" of u). Then the integrand in (3.10) is $\partial_\tau \varphi$, in particular the degree is an integer. For example $u(z) = z^d$ has degree d on the boundary of the unit disk.

It is standard to check that the degree seen as a function defined on $C^\infty(\partial\Omega, \mathbb{S}^1)$ is continuous in the C^0 norm, or in other words, the degree is preserved by homotopy. It can thus be continuously extended to an integer-valued function on the space of continuous $\mathbb{S}^1$-valued maps: this is the classical setting of degree theory (see [88] for a treatment of the classical degree theory between manifolds).

Recently, the notion of degree has been extended to certain discontinuous maps. Results of this type may be found in the work of B. White [193]. In our particular setting, Boutet-de-Monvel and Gabber (see appendix in [58]) made the crucial observation that the formula (3.10) still makes sense if $u \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, by duality between the space $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ (observe that if $u \in H^{1/2}(\partial\Omega, \mathbb{R}^2)$, then $\partial_\tau u \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$). The degree may also be defined using the Fourier coefficients of u and it is then transparent that it makes sense in $H^{1/2}$. Brezis–Nirenberg [62, 63] extended the definition of the degree to the space VMO, in any dimension.

Note that, assuming $u \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$, an alternative to (3.10) is obtained by extending u to a Sobolev map $\tilde{u} \in H^1(\Omega, \mathbb{C})$ such that $\tilde{u} = u$ on $\partial\Omega$. Then, considering the 1-form $\omega = (i\tilde{u}, \partial_1\tilde{u})dx_1 + (i\tilde{u}, \partial_2\tilde{u})dx_2$ that we write in shorthand $(i\tilde{u}, d\tilde{u})$, we have

$$\deg(u, \partial\Omega) = \frac{1}{2\pi}\int_{\partial\Omega} \omega = \frac{1}{2\pi}\int_{\Omega} d\omega.$$

But $d\omega = (id\tilde{u}, d\tilde{u}) = 2\,\mathrm{jac}\,\tilde{u}$ where $\mathrm{jac}\,\tilde{u}$ is the Jacobian determinant of $\tilde{u}$ (seen as a map from Ω to $\mathbb{R}^2$), thus

$$\deg(u, \partial\Omega) = \frac{1}{\pi}\int_{\Omega} \mathrm{jac}\,\tilde{u}(x)\,dx = \frac{1}{2\pi}\int_{\Omega} \mathrm{curl}(i\tilde{u}, \nabla\tilde{u}) \tag{3.11}$$

where, in the last expression, we have returned to our usual notation without differential forms. The definition of the degree is independent of the extension $\tilde{u}$ chosen.

If $u \in H^{1/2}(\partial\Omega, \mathbb{C})$ and $|u| > \alpha > 0$ on $\partial\Omega$, then $\deg(u, \partial\Omega)$ is defined as $\deg(u/|u|, \partial\Omega)$. If $u \in H^1(\Omega, \mathbb{C})$ satisfies $|u| > \alpha > 0$ on $\partial\Omega$, then by the trace theorem $u \in H^{1/2}(\partial\Omega, \mathbb{C})$ and the previous definition applies.

Properties

The properties of the degree for maps in $H^{1/2}(\partial\Omega, \mathbb{S}^1)$ are similar to those of the degree for smooth maps (refer to [62, 63] for proofs).

1. The degree is an integer.
2. $\deg(u, \partial\Omega)$ can be computed by (3.11) for any extension $\tilde{u} \in H^1(\Omega, \mathbb{C})$ of u.
3. For $u \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^1)$ there exists $\tilde{u} \in H^1(\Omega, \mathbb{S}^1)$ coinciding with u on $\partial\Omega$ if and only if $\deg(u, \partial\Omega) = 0$.

The last property is easily deduced from the case where Ω is the unit disk D, and from the corresponding well-known statement in the continuous setting: if $u \in C^0(\partial D, \mathbb{S}^1)$ then u can be continuously extended to $\tilde{u} \in C^0(D, \mathbb{S}^1)$ if and only if $\deg(u, \partial D) = 0$. It explains in a way why vortices need to form if a nonzero degree is prescribed on the boundary of Ω.

3.4.2 The Radial Degree-One Solution

Definition 3.6. We say u is a degree-one radial solution of

$$-\Delta u = u(1 - |u|^2) \qquad \text{in } \mathbb{R}^2 \tag{3.12}$$

if u is a solution of the form

$$u(r, \theta) = f(r) e^{i\theta}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$.

Observe that f then has to satisfy the ODE

$$-f'' - \frac{1}{r} f' + \frac{1}{r^2} f = f(1 - f^2) \qquad f(0) = 0. \tag{3.13}$$

This ODE, supplemented by the condition $f(\infty) = 1$, was studied in [111].

It holds that:

Proposition 3.11. *There exists a unique nonconstant degree-one radial solution u_0 of (3.12) such that, letting $f(r) = |u_0(r, \theta)|$, it holds that $f(r) \to 1$ as $r \to +\infty$. Moreover f is increasing and*

$$1 - f(r) \sim \frac{1}{2r^2} \quad \text{as } r \to +\infty,$$

$$\frac{1}{2} \int_{\mathbb{R}^2} \left(1 - |u_0|^2\right)^2 = \pi \tag{3.14}$$

and there exists a constant $\gamma > 0$ such that

$$\frac{1}{2} \int_{B(0,R)} |\nabla u_0|^2 + \frac{(1 - |u_0|^2)^2}{2} = \pi \log R + \gamma + o(1) \quad \text{as } R \to +\infty. \tag{3.15}$$

The existence and uniqueness of u_0 is proved in [111]. The assertion (3.14) was proved in [61] and follows from a Pohozaev type identity satisfied by the solutions in a large ball B_R together with the asymptotic behavior of f. The constant γ was introduced in [43] with a slightly different definition.

Remark 3.2. The above solution is a degree-one solution in the sense that for any R large enough, the topological degree of $\frac{u_0}{|u_0|}$ as a map from $\partial B(0,R)$ to $\mathbb{S}^1$ (as defined in Definition 3.5) is equal to 1.

This solution is also unique in the following sense:

Definition 3.7. We say u is a locally minimizing solution of $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^2$ if for any $w : \mathbb{R}^2 \to \mathbb{C}$ supported in a compact subset $K \subset \mathbb{R}^2$, we have

$$\frac{1}{2}\int_K |\nabla(u+w)|^2 + \frac{1}{2}\left(1-|u+w|^2\right)^2 \geq \frac{1}{2}\int_K |\nabla u|^2 + \frac{1}{2}\left(1-|u|^2\right)^2.$$

Theorem 3.2 (Uniqueness of locally minimizing solutions). *If u is a nonconstant solution of $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^2$ and if we make the additional assumption that either u is locally minimizing or that $\int_{\mathbb{R}^2}(1-|u|^2)^2 < +\infty$ and $\deg(u) = \pm 1$, then there exists $x_0 \in \mathbb{R}^2$ and $\theta_0 \in \mathbb{R}$ such that $u(x) = e^{i\theta_0}u_0(x-x_0)$ or $u(x) = e^{-i\theta_0}\overline{u_0}(x-x_0)$.*

This result is a combination of a theorem proved by P. Mironescu in [146] which states that solutions such that $\int_{\mathbb{R}^2}(1-|u|^2)^2 < +\infty$ and of degree one are radial; a result of I. Shafrir [186] stating that nonconstant locally minimizing solutions with $\int_{\mathbb{R}^2}(1-|u|^2)^2 < +\infty$ are of degree ± 1; and a result of E. Sandier [166] stating that locally minimizing solutions satisfy $\int_{\mathbb{R}^2}(1-|u|^2)^2 < +\infty$.

We will also use the following result on solutions on the half-plane $\mathbb{R}^2_+$:

Theorem 3.3 (Sandier [166]). *Let u be a locally minimizing solution of $-\Delta u = u(1-|u|^2)$ on $\mathbb{R}^2_+$, such that u is constant of modulus 1 on $\partial\mathbb{R}^2_+$, then u is a constant of modulus 1 on all of $\mathbb{R}^2_+$.*

3.4.3 Solutions of Higher Degree

Similarly as the radial solution of degree 1, for every $d \in \mathbb{Z}$ there exists (see [111] again) a radial solution of (3.12) with a unique zero of degree d, i.e., of the form

$$u(r,\theta) = f_d(r)e^{id\theta}$$

with f_d a real-valued function vanishing at the origin and solving an ODE analogous to (3.13):

$$-f'' - \frac{1}{r}f' + \frac{d^2}{r^2}f = f(1-f^2) \qquad f(0) = 0.$$

However, it is not known whether there are other solutions of degree $d > 1$, which would vanish in more than one point (see Open Problem 4 in Chapter 15). The only result we have is the following quantization result:

Theorem 3.4 (Brezis–Merle–Rivière [61]). *Let u be a solution of $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2}(1-|u|^2)^2 < \infty$, then*

$$\int_{\mathbb{R}^2} (1-|u|^2)^2 = 2\pi d^2$$

where $d \in \mathbb{Z}$ is the degree of $\frac{u}{|u|}$ on large enough circles.

3.5 Blow-up Limits

Definition 3.8. [Very local minimizer] Given a family of configurations $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ defined in Ω and a family of points $\{x_\varepsilon\}_\varepsilon$ in Ω, we say that $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ "very locally minimizes" G_ε around $\{x_\varepsilon\}_\varepsilon$ if for any compactly supported and smooth $w : \mathbb{R}^2 \to \mathbb{C}$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$, $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon(u_\varepsilon + w_\varepsilon, A_\varepsilon)$, where

$$w_\varepsilon(x) = w\left(\frac{x - x_\varepsilon}{\varepsilon}\right).$$

Remark 3.3. The variation on u_ε can affect the value of u_ε on the boundary of Ω, but this is allowed.

Proposition 3.12 (Behavior of blow-up limits). *Assume $h_{ex} \ll \frac{1}{\varepsilon^2}$ as $\varepsilon \to 0$, and that for every $\varepsilon > 0$ we are given a solution $(u_\varepsilon, A_\varepsilon)$ of* (3.6) *satisfying the Coulomb gauge condition and such that*

$$F_\varepsilon(u_\varepsilon, A_\varepsilon) \ll \frac{1}{\varepsilon^2} \tag{3.16}$$

as $\varepsilon \to 0$. *Then, for any family of points* $\{x_\varepsilon\}_\varepsilon$, *defining the rescaled configuration* $(v_\varepsilon, B_\varepsilon)$ *by*

$$v_\varepsilon(x) = u_\varepsilon(x_\varepsilon + \varepsilon x), \quad B_\varepsilon(x) = \varepsilon A_\varepsilon(x_\varepsilon + \varepsilon x),$$

if $\operatorname{dist}(x_\varepsilon, \partial\Omega) \gg \varepsilon$, *then after extraction of a subsequence,* $(v_\varepsilon, B_\varepsilon)$ *converges in* $C^1_{loc}(\mathbb{R}^2)$ *to* $(v, 0)$, *where* v *solves* $-\Delta v = v(1 - |v|^2)$ *in* $\mathbb{R}^2$; *if* $\operatorname{dist}(x_\varepsilon, \partial\Omega) = O(\varepsilon)$, $(v_\varepsilon, B_\varepsilon)$ *converges, after extraction of a subsequence, in* $C^1_{loc}(\mathbb{R}^2)$ *to* $(v, 0)$ *where* v *solves* $-\Delta v = v(1 - |v|^2)$ *in a half-plane with boundary condition* $\frac{\partial v}{\partial \nu} = 0$.

Moreover, if $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ *very locally minimizes* G_ε *around* $\{x_\varepsilon\}_\varepsilon$, *then in the case* $\operatorname{dist}(x_\varepsilon, \partial\Omega) \gg \varepsilon$, v *is a locally minimizing solution of (3.12), hence one of the solutions described in Theorem 3.2; in the case* $\operatorname{dist}(x_\varepsilon, \partial\Omega) = O(\varepsilon)$, v *is a constant of modulus* 1 *in the half-plane.*

The analogous result also holds for the Ginzburg–Landau equation without magnetic field:

Proposition 3.13. *The exact same result holds for solutions of* $-\Delta u_\varepsilon = \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2)$ *with Dirichlet* ($u_\varepsilon = g$ *on* $\partial\Omega$) *or Neumann boundary conditions, assuming* $E_\varepsilon(u_\varepsilon) \ll 1/\varepsilon^2$.

Note that the convergence is easily improved by bootstrapping arguments but C^1_{loc} convergence is all we will need.

Proof of Proposition 3.12. — Step 1: Convergence in the general case. Using Propositions 3.3 and 3.10, the hypothesis (3.16) implies that, letting $h_\varepsilon = \operatorname{curl} A_\varepsilon$,

$$\lim_{\varepsilon \to 0} \|\varepsilon h_\varepsilon\|_{H^1(\Omega)} = 0, \quad \lim_{\varepsilon \to 0} \|\varepsilon A_\varepsilon\|_{H^2(\Omega)} = 0.$$

In particular we find that $\varepsilon A_\varepsilon$ tend to 0 in L^∞ norm. In terms of the rescaled quantity B_ε we get

$$\lim_{\varepsilon \to 0} \|B_\varepsilon\|_{L^\infty(\Omega)} = 0. \tag{3.17}$$

Assuming for simplicity that $x_\varepsilon = 0$; in terms of the rescaled configuration $(v_\varepsilon, B_\varepsilon)$, the system (3.7) becomes

$$\begin{cases} -\Delta v_\varepsilon + 2i(B_\varepsilon \cdot \nabla) v_\varepsilon = v_\varepsilon(1 - |v_\varepsilon|^2) - |B_\varepsilon|^2 v_\varepsilon & \text{in } \Omega/\varepsilon \\ -\Delta B_\varepsilon = \varepsilon^2 (iv_\varepsilon, \nabla v_\varepsilon - iB_\varepsilon v_\varepsilon) & \text{in } \Omega/\varepsilon \\ \operatorname{curl} B_\varepsilon = \varepsilon^2 h_{\text{ex}} & \text{outside } \Omega/\varepsilon \\ \nu \cdot \nabla_{B_\varepsilon} v_\varepsilon = 0 & \text{on } \partial\Omega/\varepsilon. \end{cases} \tag{3.18}$$

We now invoke elliptic regularity for the first equation. From (3.17) and Proposition 3.9, the right-hand side is bounded in L^∞ and we may apply L^p estimates (see for instance [100] Theorem 9.11) to find that for any fixed ball B_R, the family $\{v_\varepsilon\}_\varepsilon$ is bounded in $W^{2,p}(B_R)$, for any $p > 1$, hence in $C^{1,\alpha}$ for any $0 < \alpha < 1$. From the compactness of the embedding of $C^{1,\alpha}$ into C^1, and using larger and larger balls together with a diagonal argument, we may extract a subsequence, still denoted $\{\varepsilon\}$, such that $\{v_\varepsilon\}_\varepsilon$ converges locally in C^1 norm to some v.

The right-hand side of $-\Delta B_\varepsilon = \varepsilon^2(iv_\varepsilon, \nabla v_\varepsilon - iB_\varepsilon v_\varepsilon)$ is now known to be bounded in L^∞. Then from (3.17) and elliptic regularity we find as above that after extraction of a subsequence, $\{B_\varepsilon\}_\varepsilon$ converges locally in C^1 norm to some B. From (3.17) the limit B is necessarily 0.

Passing to the limit in the equations we find that v solves $-\Delta v = v(1-|v|^2)$.

— *Step 2: proof of the last assertion.* We prove that if the solutions are very local minimizers, then their limit v is locally minimizing. Assume $w : \mathbb{R}^2 \to \mathbb{C}$ is supported in a compact subset $K \subset \mathbb{R}^2$. Scale back w to define $w_\varepsilon(x) = w\left((x - x_\varepsilon)/\varepsilon\right)$.

We define

$$F(u) = \int_K |\nabla u|^2 + \frac{1}{2}\left(1 - |u|^2\right)^2$$

$$F_\varepsilon(u) = \int_K |\nabla_{B_\varepsilon} u|^2 + \frac{1}{\varepsilon^2}(\operatorname{curl} B_\varepsilon - \varepsilon^2 h_{\rm ex})^2 + \frac{1}{2}\left(1 - |u|^2\right)^2.$$

From the C^1 convergence of v_ε and B_ε, and the relations (3.18), it is easy to check that

$$F(v + w) - F(v) = \lim_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon + w) - F_\varepsilon(v_\varepsilon).$$

But the right-hand side is equal to $G_\varepsilon(u_\varepsilon + w_\varepsilon, A_\varepsilon) - G_\varepsilon(u_\varepsilon, A_\varepsilon)$ and is therefore positive for ε small enough since $\{u_\varepsilon\}_{\varepsilon > 0}$ is very locally minimizing around $\{x_\varepsilon\}_\varepsilon$. It follows that $F(v + w) - F(v) \geq 0$ and the proposition is proved.

For the case where $\operatorname{dist}(x_\varepsilon, \partial\Omega) \leq C\varepsilon$, it is easy to see that up to translation and extraction, B_ε converges to 0 as before, and v_ε converges to a solution of $-\Delta v = v(1 - |v|^2)$ on the half-plane $\mathbb{R}^2_+$ with boundary condition $\frac{\partial v}{\partial \nu} = 0$.

Reflecting that solution with respect to $\partial\mathbb{R}^2_+$ yields a solution with degree 0 on all of $\mathbb{R}^2$. Moreover, by the same arguments as above, this solution is locally minimizing in $\mathbb{R}^2$, hence by the result of Sandier [165] mentioned above, it satisfies $\int_{\mathbb{R}^2}(1-|v|^2)^2 < \infty$. Then, from the result of [61], such a solution with total degree 0 has to be constant: its modulus is constant equal to 1 from Theorem 3.4, which implies that $\Delta v = 0$, and thus that v, being harmonic and bounded in $\mathbb{R}^2$, is constant. □

Proof of Proposition 3.13. The proof in the Neumann case is identical to the one above. The proof in the Dirichlet case follows exactly the same lines. The only difference is that if $\mathrm{dist}(x_\varepsilon, \partial\Omega) = O(\varepsilon)$, the limiting v is a solution of $-\Delta v = v(1-|v|^2)$ in the half-plane $\mathbb{R}^2_+$ with $|v| = 1$ on the boundary of the half-plane. In the case of very local minimizers, the result of Theorem 3.3 allows us to conclude that v is also constant. □

Remark 3.4. In the case where the solutions are not very local minimizers, the limit v can a priori be any solution of (3.12) as described in Section 3.4. It can be one of the solutions such that $\int(1-|u|^2)^2 < \infty$ considered in Theorem 3.4. It can also be a solution that does not satisfy this condition, such as the solution $u \equiv 0$. This can be achieved by considering solutions with a unique zero of degree $d_\varepsilon \to \infty$ as $\varepsilon \to 0$ (such as in [30]): $f_{d_\varepsilon}(r/\varepsilon)e^{id_\varepsilon\theta}$ in the notation of Section 3.4.3, for which $f_{d_\varepsilon} \to 0$ a.e. as $\varepsilon \to 0$.

Corollary 3.1. *If $(u_\varepsilon, A_\varepsilon)$ is a solution of (3.6) such that $F_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) \ll \frac{1}{\varepsilon^2}$, then there exists a constant C such that*

$$|\nabla_{A_\varepsilon} u_\varepsilon| \leq \frac{C}{\varepsilon}.$$

Proof. Since $|\nabla_A u|$ is a gauge-invariant quantity, we may assume that we are in the Coulomb gauge. Using the same notation as above, we have $\varepsilon|\nabla_{A_\varepsilon} u_\varepsilon| = |\nabla_{B_\varepsilon} v_\varepsilon| \to |\nabla v|$ in view of the C^1_{loc} convergence of $(v_\varepsilon, B_\varepsilon)$. We deduce that $|\nabla_A u|(x) \leq \frac{C}{\varepsilon}$, where we claim that C is bounded independently of the point and of the solution. If it were not, then we could find a sequence of solutions $(u_\varepsilon, A_\varepsilon)$ of (3.6) and a sequence of points x_ε such that $\varepsilon|\nabla_{A_\varepsilon} u_\varepsilon|(x_\varepsilon) \to +\infty$. Arguing as in the proof of Proposition 3.12, we would find that, up to extraction, $(v_\varepsilon, B_\varepsilon)$ converges in C^1_{loc} to some $(v, 0)$ and thus $\varepsilon|\nabla_{A_\varepsilon} u_\varepsilon| = |\nabla_{B_\varepsilon} v_\varepsilon|$ is convergent, and this would contradict the assumption. □

Remark 3.5. Adjusting the proof of Proposition 3.12, we can easily show that, if we only assume $F_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) \le \frac{C}{\varepsilon^2}$ and $h_{\rm ex} \le \frac{C}{\varepsilon^2}$, then we still have $|\nabla_{A_\varepsilon} u_\varepsilon| \le \frac{C}{\varepsilon^2}$.

Bibliographic notes on Chapter 3: The material of the first parts of this chapter (existence of minimizers, derivation of the equations, regularity) is fairly standard. For the case with magnetic field, one may refer to Bethuel–Rivière [52] and references therein, also to the survey paper of Du–Gunzburger–Peterson [85]; and for the case without magnetic field to Bethuel–Brezis–Hélein [42]. For solutions in the plane the most important references are (in chronological order) Hervé-Hervé [111], Brezis–Merle–Rivière [61] and Mironescu [142]. Results on blow-up of solutions can be found in a scattered way in the literature on the functional without magnetic field (see, e.g., the works of Bethuel–Brezis–Hélein and Comte–Mironescu). We included here a version more specific to the case with magnetic field, including the possibility of very large energies and the notion of very local minimizers.

Chapter 4

The Vortex-Balls Construction

The aim of this chapter is to provide one of the basic tools for the analysis of the Ginzburg–Landau functional in terms of vortices.

When studying critical fields in $\mathbb{R}^2$, we have constructed an approximate vortex for which u had a zero of degree one. The energy of this solution was approximately the sum of a term depending on $h_{\rm ex}$ and $\pi|\log\varepsilon|$. The former represents the interaction of the vortex with the applied field and the latter — although we did not explicitly state this result — corresponds to the *free energy* of the vortex, i.e., its energy when $h_{\rm ex}$ is taken to be zero.

Given an *arbitrary* configuration (u, A), we will show that one can describe it energetically as a collection of vortices glued together, as long as its Ginzburg–Landau energy is not extremely large, but without assuming that it solves any equation. More precisely we construct a set of disjoint balls of sufficiently small radii (how small depends on which construction) which cover the "bad set" where $|u|$ is smaller than some threshold < 1, hence which contains the zero-set of u and all possible vortices. Moreover, each ball B will contain an amount of energy at least of (typically) $\pi|d|\log\frac{r}{\varepsilon}$ where $d = \deg(u/|u|, \partial B)$, and r is the radius of B.

The construction requires only a weak control on the energy of u, essentially it must be $\ll \frac{1}{\varepsilon}$. This is much larger than the energy of one vortex which is approximately $\pi|\log\varepsilon|$, hence it allows a number of vortices which is unbounded as $\varepsilon \to 0$. It uses a ball growing argument which was introduced independently in [113] and [166]; we give here a presentation close to [166, 170], with sharper estimates. All the results can be used for the functional without magnetic field (1.2) simply by setting the magnetic potential A to 0.

The main result of the chapter, namely Theorem 4.1 will be used repeatedly in Chapters 7 to 12, which deal with the minimization of the Ginzburg–Landau functional, but its proof can safely be skipped to read them. Chapter 5 only uses the ball-growth mechanism described in the second section of the chapter, and Chapter 13 uses only Propositions 4.3 and 4.8.

The most delicate part of the chapter is the proof of Proposition 4.7, which occupies the last section. The refinements there are motivated by the fact that we need to obtain the optimal error term, namely a constant times the total degree, for later applications. This aside, the proof of Theorem 4.1 is essentially contained in sections 2–4.

4.1 Main Result

Here and in the rest of this chapter, Ω is an open subset of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ are C^1.

We recall that if $u : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$,

$$F_\varepsilon(u, A, \Omega) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}. \tag{4.1}$$

For any function $\rho : \Omega \to \mathbb{R}$ we set

$$F_\varepsilon(\rho, \Omega) = \frac{1}{2} \int_\Omega |\nabla \rho|^2 + \frac{(1 - \rho^2)^2}{2\varepsilon^2}.$$

Note that the notation above is consistent in the sense that if $u = \rho$ is real-valued, then $F_\varepsilon(\rho, \Omega) = F_\varepsilon(u, 0, \Omega)$.

We denote

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}. \tag{4.2}$$

If $\mathcal{B}$ is a collection of balls, then $r(\mathcal{B})$ denotes the sum of the radii of the balls in the collection.

Theorem 4.1 (Lower bound through the ball-construction). *For any $\alpha \in (0, 1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if (u, A) is a configuration such that $F_\varepsilon(|u|, \Omega) \le \varepsilon^{\alpha-1}$, the following holds.*

For any $1 > r > C\varepsilon^{\alpha/2}$, where C is a universal constant, there exists a finite collection of disjoint closed balls $\mathcal{B} = \{B_i\}_{i\in I}$ such that

1. *$r(\mathcal{B}) = r$.*

2. *Letting $V = \Omega_\varepsilon \cap \cup_{i\in I} B_i$,*

$$\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \geq \varepsilon^{\frac{\alpha}{4}}\} \subset V.$$

3. *Writing $d_i = \deg(u, \partial B_i)$, if $B_i \subset \Omega_\varepsilon$ and $d_i = 0$ otherwise,*

$$\frac{1}{2}\int_V \left[|\nabla_A u|^2 + r^2|\operatorname{curl} A|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right] \geq \pi D\left(\log\frac{r}{D\varepsilon} - C\right), \tag{4.3}$$

where $D = \sum_{i\in I}|d_i|$ is assumed to be nonzero and C is a universal constant.

4. *If the stronger assumption $F_\varepsilon(u, A, \Omega) \leq \varepsilon^{\alpha-1}$ holds, then*

$$D \leq C\frac{F_\varepsilon(u, A, \Omega)}{\alpha|\log\varepsilon|}, \tag{4.4}$$

where C is a universal constant.

Finally, if $1 > r_1 > r_2 > \varepsilon^{\alpha/2}$ and $\mathcal{B}_1$, $\mathcal{B}_2$ are the corresponding families of balls, then every ball in $\mathcal{B}_2$ is included in one of the balls of $\mathcal{B}_1$.

Remark 4.1. The term $\log\frac{r}{D\varepsilon}$ in (4.3) is optimal, as can be seen by taking D identical vortices of degree 1; the total radius of the balls being r, we can expect D final balls of radius r/D each, each containing an energy $\pi\log\frac{r/D}{\varepsilon}$.

The proof of this theorem will occupy the rest of this chapter.

4.2 Ball Growth

In essence, the theorem is proved by adding up lower bounds for the energy of (u, A) on annuli which avoid the set where $|u|$ is different from 1. For these lower bounds to add up, they need to be computed on conformally identical annuli, and we describe here the tool which allows us to do this.

Notation: If B is a ball, $r(B)$ denotes its radius. If $\mathcal{B}$ is a collection of balls, then $r(\mathcal{B})$ is the sum of the radii of the balls in the collection. For

$\lambda \geq 0$ the ball λB is the ball with same center as B and radius multiplied by λ. If $\mathcal{B}$ is a collection of balls, then $\lambda\mathcal{B} = \{\lambda B \mid B \in \mathcal{B}\}$. With an abuse of notation, we will also write $\int_{\mathcal{B}}$ to denote $\int_{\cup_{B\in\mathcal{B}} B}$, write $\mathcal{B} \cap U$ to denote the collection $\{B \cap U\}_{B\in\mathcal{B}}$, and $U\backslash\mathcal{B}$ to denote $U\backslash(\cup_{B\in\mathcal{B}} B)$.

Theorem 4.2 (Ball growth). *Let $\mathcal{B}_0$ be a finite collection of disjoint closed balls. There exists a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ of collections of disjoint closed balls such that $\mathcal{B}(0) = \mathcal{B}_0$ and*

1. *For every $s \geq t \geq 0$,*
$$\bigcup_{B\in\mathcal{B}(t)} B \subset \bigcup_{B\in\mathcal{B}(s)} B.$$
2. *There exists a finite set $T \subset \mathbb{R}_+$ such that if $[t_0, t_1] \subset \mathbb{R}_+ \setminus T$, then $\mathcal{B}(t_1) = e^{t_1 - t_0}\mathcal{B}(t_0)$.*
3. *$r(\mathcal{B}(t)) = e^t r(\mathcal{B}_0)$ for every $t \in \mathbb{R}_+$.*

Lemma 4.1 (Merging). *Assume B_1 and B_2 are closed balls in $\mathbb{R}^n$ such that $B_1 \cap B_2 \neq \varnothing$. Then there exists a closed ball B such that $r(B) = r(B_1) + r(B_2)$ and $B_1 \cup B_2 \subset B$.*

Proof. If $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$, let
$$B = B\left(\frac{r_1 a_1 + r_2 a_2}{r_1 + r_2}, r_1 + r_2\right). \qquad \square$$

Proof of the theorem. We first perform *growing*, starting from $\mathcal{B}_0$. We let $\mathcal{B}(t) = e^t\mathcal{B}_0$ for every $t \geq 0$ and let t_0 be the supremum of the times such that $\mathcal{B}(t)$ is a collection of disjoint closed balls. If $t_0 = +\infty$, we are done.

If not, then the balls in $\mathcal{B}(t_0)$ have disjoint interiors but some have intersecting closures. Then we perform *merging*. Assume $B_1, B_2 \in \mathcal{B}(t_0)$ have intersecting closures and call r_1, r_2 their radii. Then we group them into a larger ball B with radius $r = r_1 + r_2$ using Lemma 4.1. We then remove B_1, B_2 from the collection $\mathcal{B}(t_0)$ and add to it B. Repeating this operation enough times, we get a family $\mathcal{B}'(t_0)$ of balls with nonintersecting closures. Moreover $r(\mathcal{B}(t_0)) = r(\mathcal{B}'(t_0))$ and $\cup_{B\in\mathcal{B}(t_0)} B \subset \cup_{B\in\mathcal{B}'(t_0)} B$. Finally $\mathcal{B}'(t_0)$ contains strictly fewer balls than $\mathcal{B}(t_0)$. We then define $\mathcal{B}(t_0) = \mathcal{B}'(t_0)$, and perform growing starting from $\mathcal{B}(t_0)$.

We may repeat this process to define a family $\mathcal{B}(t)$ of disjoint closed balls for every $t \geq 0$. Indeed the merging process can occur only a finite number of times since it strictly decreases the number of balls in the collection which was finite to begin with. Property 1 is clearly satisfied. If we define T to be the set of times at which merging occurs, Property 2 is satisfied as well. Property 3 is obvious for $t = 0$ and is clearly preserved during growing and merging, hence it is true for all $t \geq 0$. □

Remark 4.2. We cannot ensure uniqueness in this construction because there is a choice in the order in which we merge the balls if there are more than two intersecting balls at a given t.

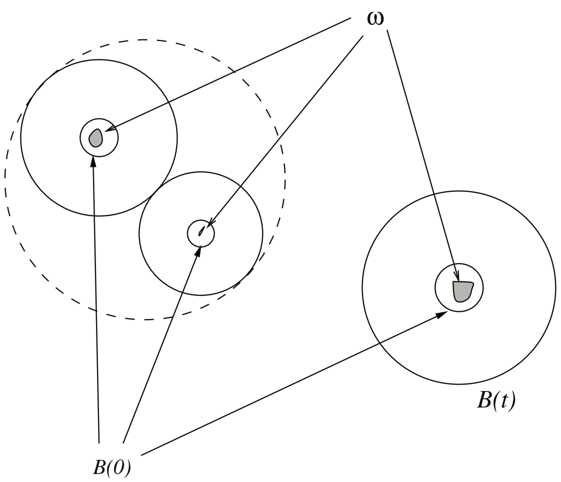

Figure 4.1: Ball growth starting from an initial set ω.

Additional properties of this construction follow.

Definition 4.1. [and notation] Let $\mathcal{F}(x,r)$ be a function defined on $\mathbb{R}^2 \times \mathbb{R}_+$. We will also see $\mathcal{F}$ as a function defined on the set of all closed balls, and write $\mathcal{F}(B)$ for $\mathcal{F}(x,r)$ if $B = B(x,r)$. We will also write $\mathcal{F}(\mathcal{B})$ as a shorthand for $\sum_{B\in\mathcal{B}} \mathcal{F}(B)$ if $\mathcal{B}$ is a collection of balls.

We say that $\mathcal{F}$ is monotonic if $\mathcal{F}$ is continuous with respect to r and for any families of disjoint closed balls $\mathcal{B}, \mathcal{B}'$ such that $\cup_{B\in\mathcal{B}} B \subset \cup_{B\in\mathcal{B}'} B$

$$\mathcal{F}(\mathcal{B}) \leq \mathcal{F}(\mathcal{B}').$$

This implies, in particular, that $\mathcal{F}$ is nondecreasing in r.

Proposition 4.1. *Let $\mathcal{F}:\mathbb{R}^2\times\mathbb{R}_+\to\mathbb{R}_+$ be monotonic in the above sense. Let $\mathcal{B}_0$ be a finite collection of disjoint closed balls and $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ satisfying the results of Theorem 4.2, then, for every $s\ge 0$,*

$$\mathcal{F}(\mathcal{B}(s))-\mathcal{F}(\mathcal{B}_0)\ge\int_{t=0}^{s}\sum_{B(x,r)\in\mathcal{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt,\tag{4.5}$$

and for every $B\in\mathcal{B}(s)$, we have

$$\mathcal{F}(B)-\mathcal{F}(\mathcal{B}_0\cap B)\ge\int_{t=0}^{s}\sum_{B(x,r)\in\mathcal{B}(t)\cap B} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt.\tag{4.6}$$

Remark 4.3. If $\mathcal{F}$ is C^1 with respect to r, then (4.5) clearly makes sense. If $\mathcal{F}$ is only continuous, then the integral still makes sense if we see $\frac{\partial\mathcal{F}}{\partial r}(x,.)$ as a measure, which we can do since $\mathcal{F}$ is monotonic with respect to r. Note that in this case, and since $\mathcal{F}$ is continuous with respect to r, this measure has no atoms and therefore the meaning of $\int_a^b\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt$ is unambiguous; that is, does not depend on whether the endpoints are included or excluded.

Proof. Let T be the finite set of Theorem 4.2. Then $(0,s)\setminus T$ may be written as a disjoint union $\cup_{i=1}^k(s_{i-1},s_i)$, where $s_0=0$ and $s_k=s$. Writing $\mathcal{B}(t)=\{B_1(t),\dots,B_n(t)\}$ we have $B_i(t)=e^{t-s_0}B_i(0)$ for $t\in[s_0,s_1)$. Letting $B_i(t)=B(x_i,r_i(t))$ we thus have $r_i'(t)=r_i(t)$ and then

$$\frac{d}{dt}\mathcal{F}(\mathcal{B}(t))=\sum_{i=1}^n\frac{d}{dt}\mathcal{F}(x_i,r_i(t))=\sum_{i=1}^n r_i(t)\frac{\partial}{\partial r}\mathcal{F}(x_i,r_i(t)).$$

Integrating on (s_0,s_1) we find

$$\mathcal{F}(\mathcal{B}(s_1))^- -\mathcal{F}(\mathcal{B}(s_0))=\int_{t=s_0}^{s_1}\sum_{B(x,r)\in\mathcal{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt,$$

where we have written $\mathcal{F}(\mathcal{B}(s_1))^-$ for the limit of $\mathcal{F}(\mathcal{B}(t))$ as t increases to s_1. By the monotonicity of $\mathcal{F}$, this is smaller than $\mathcal{F}(\mathcal{B}(s_1))$ hence

$$\mathcal{F}(\mathcal{B}(s_1))-\mathcal{F}(\mathcal{B}(s_0))\ge\int_{t=s_0}^{s_1}\sum_{B(x,r)\in\mathcal{B}(t)} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt.$$

Repeating this in every interval (s_{i-1}, s_i) and summing yields the result (4.5).

Let now B be a ball in $\mathcal{B}(s)$. Observe that from assertion 1 of Theorem 4.2 that for $t \le s$, the balls in $\mathcal{B}(t)$ found before are either included in B, or do not intersect B. Starting from the initial collection $\mathcal{B}_0 \cap B$, we get that for $t \le s$, the collection $\mathcal{B}(t) \cap B$ still satisfies the results of Theorem 4.2 (in other words, we can redo the construction starting from the initial collection $\mathcal{B}_0 \cap B$ and obtain the collection $\mathcal{B}(t) \cap B$).

We may then apply the result (4.5) with this new restricted family. It yields

$$\mathcal{F}(\mathcal{B}(s) \cap B) - \mathcal{F}(\mathcal{B}_0 \cap B) \ge \int_{t=0}^{s} \sum_{B(x,r) \in \mathcal{B}(t) \cap B} r \frac{\partial \mathcal{F}}{\partial r}(x, r)\, dt.$$

But for $t = s$, the only ball in the new collection $\mathcal{B}(s) \cap B$ is B, hence by definition $\mathcal{F}(\mathcal{B}(s) \cap B) = \mathcal{F}(B)$ and (4.6) is proved. $\square$

4.3 Lower Bounds for $\mathbb{S}^1$-valued Maps

The construction of the previous section allows us to obtain a result very similar to Theorem 4.1 if we assume $|u| = 1$.

Notation: We let

$$\nabla_A u = \nabla u - iAu, \quad \partial_v^A u = v \cdot \nabla u - i(A \cdot v)u,$$

where v is a vector.

For a bounded domain $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{C}$ we let

$$E(u, \Omega) = \frac{1}{2} \int_\Omega |\nabla u|^2.$$

If $A : \Omega \to \mathbb{R}^2$, we let

$$E_A(u, \Omega) = \frac{1}{2} \int_\Omega |\nabla_A u|^2, \tag{4.7}$$

and

$$H(A, \Omega) = \frac{1}{2} \int_\Omega |\operatorname{curl} A|^2. \tag{4.8}$$

Also, given a map u defined in Ω and a ball B such that u does not vanish on ∂B, we let

$$d_B = \begin{cases} \deg(u, \partial B) & \text{if } B \subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

For the definition of the degree, see Definition 3.5.

We begin with a simple lemma.

Lemma 4.2. *Assume Ω is an open subset of $\mathbb{R}^2$ and ω a compact subset of Ω. Assume $v : \Omega \setminus \omega \to \mathbb{S}^1$ is C^1. If $\mathcal{B}$, $\mathcal{B}'$ are two finite collections of disjoint closed balls such that $\omega \subset \cup_{B\in\mathcal{B}} B$ and $\cup_{B\in\mathcal{B}} B \subset \cup_{B'\in\mathcal{B}'} B'$, then*

$$\sum_{B\in\mathcal{B}} |d_B| \geq \sum_{B'\in\mathcal{B}'} |d_{B'}|.$$

Proof. First note that under our assumptions, every ball in $\mathcal{B}$ is included in one and only one ball of $\mathcal{B}'$. Then for every $B' \in \mathcal{B}'$ such that $B' \subset \Omega$

$$\deg(v, \partial B') = \sum_{\substack{B\in\mathcal{B} \\ B\subset B'}} \deg(v, \partial B).$$

Taking absolute values and summing over balls $B' \in \mathcal{B}'$ such that $B' \subset \Omega$ proves the lemma. □

The lower bound which is used later on is Proposition 4.3 rather than the following one, simply bounding the Dirichlet energy of $\mathbb{S}^1$-valued maps. However, this one is included because it illustrates the method without being obscured by technicalities.

Proposition 4.2. *Assume Ω is an open subset of $\mathbb{R}^2$ and $\mathcal{B}_0$ is a finite collection of disjoint closed balls. Let $\omega = \cup_{B\in\mathcal{B}_0} B$ and let $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ be defined by Theorem 4.2. Then for any $v : \Omega \setminus \omega \to \mathbb{S}^1$ in C^1 and any $s \geq 0$, for every $B \in \mathcal{B}(s)$, we have*

$$E\left(v, (B \cap \Omega) \setminus \omega\right) \geq \pi \int_0^s \|D_B\|^2(t)\, dt \tag{4.9}$$

and

$$E\left(v, (B \cap \Omega) \setminus \omega\right) \geq \pi |d_B| \log \frac{r_1}{r_0}, \tag{4.10}$$

where $r_0 = r(\mathcal{B}_0)$, $r_1 = r(\mathcal{B}(s)) = e^s r_0$, *and*

$$\|D_B\|^2(t) = \sum_{B' \in \mathcal{B}(t) \cap B} d_{B'}^2.$$

Proof. We define

$$\mathcal{F}(x,r) = \frac{1}{2} \int\limits_{B(x,r)\cap\Omega} |\nabla v|^2.$$

We have

$$\frac{\partial \mathcal{F}}{\partial r}(x,r) = \frac{1}{2} \int\limits_{\partial B(x,r)\cap\Omega} |\nabla v|^2 \tag{4.11}$$

and the crucial inequality (which we prove below):

Lemma 4.3. *For any* $v : \partial B \to \mathbb{S}^1$, *where* B *is a ball of radius* r *in* $\mathbb{R}^2$, *we have*

$$\frac{1}{2} \int\limits_{\partial B} |\nabla v|^2 \geq \pi \frac{d^2}{r}, \tag{4.12}$$

where $d = \deg(v, \partial B)$.

We now apply Proposition 4.1. Inserting (4.12) and (4.11) into (4.6) we find, for every $B \in \mathcal{B}(s)$,

$$\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0) \cap B) \geq \pi \int\limits_{t=0}^{s} \sum_{B' \in \mathcal{B}(t) \cap B} d_{B'}^2 \, dt, \tag{4.13}$$

proving (4.9). But

$$\sum_{B' \in \mathcal{B}(t) \cap B} d_{B'}^2 \geq \sum_{B' \in \mathcal{B}(t) \cap B} |d_{B'}|,$$

and from Lemma 4.2, using the fact that $\mathcal{B}(s) \cap B = \{B\}$,

$$\sum_{B' \in \mathcal{B}(t) \cap B} |d_{B'}| \geq \sum_{B' \in \mathcal{B}(s) \cap B} |d_{B'}| = |d_B|.$$

Replacing this in (4.13) yields $\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0) \cap B) \geq \pi |d_B| s$. Since $s = \log(r_1/r_0)$, and since

$$\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0) \cap B) = E(v, B \cap \Omega) - \sum_{\substack{B' \in \mathcal{B}_0 \\ B' \subset B}} E(v, B' \cap \Omega) \leq E\left(v, (B \cap \Omega) \setminus \omega\right),$$

the inequality (4.10) is proved. □

Proof of Lemma 4.3. Let (τ, ν) be respectively a unit tangent and unit normal vector to ∂B. Then

$$|\nabla v|^2 = |\partial_\nu v|^2 + |\partial_\tau v|^2$$

therefore,

$$\frac{1}{2} \int_{\partial B} |\nabla v|^2 \geq \frac{1}{2} \int_{\partial B} |\partial_\tau v|^2.$$

But

$$\int_{\partial B} |\partial_\tau v| \geq \left| \int_{\partial B} (iv, \partial_\tau v) \right| = 2\pi |d|,$$

thus, using the Cauchy–Schwarz inequality,

$$\frac{1}{2} \int_{\partial B} |\nabla v|^2 \geq \pi \frac{|d|^2}{r},$$

which proves the lemma. □

We also prove the following variant of Proposition 4.2 which includes the magnetic potential A.

Proposition 4.3. *Assume Ω is an open subset of $\mathbb{R}^2$ and $\mathcal{B}_0$ is a finite collection of disjoint closed balls. Let $\omega = \cup_{B \in \mathcal{B}_0} B$ and let $\{\mathcal{B}(t)\}_{t \in \mathbb{R}_+}$ be defined by Theorem 4.2. Then for any $v : \Omega \setminus \omega \to \mathbb{S}^1$ and any $A : \Omega \to \mathbb{R}^2$ in C^1, for any $s \geq 0$ such that $r(\mathcal{B}(s)) \leq 1$, and for any $B \in \mathcal{B}(s)$,*

$$E_A\left(v, (B \cap \Omega) \setminus \omega\right) + r_1(r_1 - r_0) H(A, B \cap \Omega) \geq \pi \int_{t=0}^{s} \|D_B\|^2(t) \left(1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)}\right) dt. \quad (4.14)$$

and

$$E_A\left(v,(B\cap\Omega)\setminus\omega\right)+r_1(r_1-r_0)H(A,B\cap\Omega)$$
$$\geq \pi|d_B|\left(\log\frac{r_1}{r_0}-\log 2\right), \quad (4.15)$$

where $r_0 = r(\mathcal{B}_0)$, $r_1 = r(\mathcal{B}(s)) = e^s r_0$, *and*

$$\|D_B\|^2(t) = \sum_{B'\in\mathcal{B}(t)\cap B} d_{B'}^2.$$

The proof relies on:

Lemma 4.4. *If B is a ball of radius r in $\mathbb{R}^2$, then for any $v : \partial B \to \mathbb{S}^1$, any $A : B \to \mathbb{R}^2$ in C^1, and any $\lambda > 0$,*

$$\frac{1}{2}\int_{\partial B}|\nabla_A v|^2 + \frac{\lambda}{2}\int_B(\operatorname{curl} A)^2 \geq \pi\frac{|d_B|^2}{r}\left(\frac{1}{1+\frac{r}{2\lambda}}\right). \quad (4.16)$$

Proof of Lemma 4.4. Let

$$X = \int_B \operatorname{curl} A.$$

Writing $v = e^{i\varphi}$, choosing the right orientation for the unit vector τ tangent to ∂B, from Stokes's formula, we have

$$\int_{\partial B}\tau\cdot(\nabla\varphi - A) = 2\pi d_B - X.$$

From Lemma 3.4 and since $|v| = 1$ on ∂B we have $|\tau\cdot(\nabla\varphi - A)| \leq |\nabla_A v|$ on ∂B. Then the Cauchy–Schwarz inequality yields

$$\frac{1}{2}\int_{\partial B}|\nabla_A v|^2 \geq \frac{1}{2}\frac{(2\pi d_B - X)^2}{2\pi r}. \quad (4.17)$$

On the other hand, by Cauchy–Schwarz again,

$$\frac{\lambda}{2}\int_B|\operatorname{curl} A|^2 \geq \frac{\lambda}{2}\frac{X^2}{\pi r^2}. \quad (4.18)$$

Summing (4.17) and (4.18) and minimizing with respect to X yields (4.16). ∎

Proof of Proposition 4.3. We define

$$\mathcal{F}(x,r) = E_A(v, B(x,r)\cap\Omega) + r(r_1 - r_0)H(A, B(x,r)\cap\Omega).$$

We apply Proposition 4.1 with $\mathcal{F}$ defined above. We have

$$\frac{\partial\mathcal{F}}{\partial r}(x,r) \ge \frac{1}{2}\int_{\partial B(x,r)\cap\Omega} |\nabla_A v|^2 + \frac{r_1 - r_0}{2}\int_{\Omega\cap B(x,r)} (\operatorname{curl} A)^2. \tag{4.19}$$

Inserting (4.16) with $\lambda = r_1 - r_0$ and (4.19) into (4.6) we find that for every $B \in \mathcal{B}(s)$,

$$\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0)\cap B) \ge \pi \int_{t=0}^{s} \sum_{B'\in\mathcal{B}(t)\cap B} d_{B'}^2 \left(\frac{1}{1 + \frac{r(B')}{2\lambda}}\right) dt.$$

But $1/(1+x) \ge 1 - x$ if $x \ge 0$, hence for $t \in (0,s)$ and every $B' \in \mathcal{B}(t)$,

$$\frac{1}{1 + \frac{r(B')}{2\lambda}} \ge 1 - \frac{r(B')}{2\lambda} \ge 1 - \frac{r(\mathcal{B}(t))}{2\lambda} = 1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)}.$$

We deduce (4.14). Moreover, using Lemma 4.2,

$$\sum_{B'\in\mathcal{B}(t)\cap B} d_{B'}^2 \ge \sum_{B'\in\mathcal{B}(t)\cap B} |\deg(v, \partial B')| \ge \sum_{B'\in\mathcal{B}(s)\cap B} |d_{B'}| = |d_B|.$$

If $\log\frac{r_1}{r_0} \le \log 2$, then the desired inequality is trivially true. If not, then $r_1 \ge 2r_0$ and then, since $r(\mathcal{B}(t)) \le r_1$, we always have $1 - \frac{r(\mathcal{B}(t))}{2(r_1 - r_0)} \ge 0$. Therefore, replacing with the previous relation in (4.14), we are led to

$$\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0)\cap B) \ge \pi|d_B| \int_{t=0}^{s} \left(1 - \frac{r(\mathcal{B}(t))}{2\lambda}\right) dt.$$

But from Theorem 4.2, item 3, the antiderivative of $r(\mathcal{B}(t))$ is itself. Hence the integral of $r(\mathcal{B}(t))$ over $[0,s]$ is equal to $r_1 - r_0$, which is equal to λ and the above reduces to

$$\mathcal{F}(B) - \mathcal{F}(\mathcal{B}(0)\cap B) \ge \pi|d_B|(s - 1/2).$$

Since $s = \log(r_1/r_0)$ and replacing as in the proof of Proposition 4.2, the inequality (4.15) follows. □

4.4 Reduction to $\mathbb{S}^1$-valued Maps

The proof of Theorem 4.1 can be reduced to proving a similar lower bound for $u/|u|$ on various sets. In this section, we will state this proposition precisely and show how Theorem 4.1 follows from it. The proof of the proposition itself will occupy the remaining sections of this chapter.

First we need to introduce the following notion.

4.4.1 Radius of a Compact Set

Definition 4.2. The *radius* of a compact set $\omega \subset \mathbb{R}^2$ is the infimum over all finite coverings of ω by closed balls $B_1, \dots, B_k$ of $r(B_1) + \cdots + r(B_k)$. We write it $r(\omega)$.

Remark 4.4. 1) Note that in this definition we may assume the covering is by disjoint balls. Indeed if B_1 and B_2 satisfy $B_1 \cap B_2 \neq \varnothing$, then Lemma 4.1 ensures the existence of a ball B such that $B_1 \cup B_2 \subset B$ and $r(B) = r(B_1) + r(B_2)$. Using this to group together intersecting balls, a finite covering may be replaced by a covering by disjoint balls leaving the sum of radii unchanged.

2) Clearly, if $A \subset B$, then $r(A) \leq r(B)$.

3) The infimum which defines the radius is not necessarily achieved.

There is a relationship between radius and perimeter:

Proposition 4.4. *Assume ω is a compact subset of $\mathbb{R}^2$. Then $2r(\omega) \leq \mathcal{H}^1(\partial\omega)$, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.*

Proof. By definition of the Hausdorff measure, it suffices to show that if $\{B_i\}_{i\in\mathbb{N}}$ is any covering of $\partial\omega$ by open balls, then $r(\omega) \leq \sum_i r(B_i)$. Since $\partial\omega$ is compact it suffices to work with a finite covering, and then taking the closures and using Lemma 4.1, we may assume the balls are closed and disjoint. In particular $A = \mathbb{R}^2 \setminus \cup_{i=1}^k B_i$ is connected. Now if $B_1, \dots, B_k$ cover $\partial\omega$, we claim they cover ω and therefore $r(\omega) \leq \sum_i r(B_i)$, from which the result follows by definition of the Hausdorff measure. The claim follows by noting that A—which is connected—intersects the complement of ω because ω is bounded. Thus, if A intersected ω it would also intersect $\partial\omega$, which is impossible from the definition of A. Thus $\omega \subset \mathbb{R}^2 \setminus A = \cup_{i=1}^k B_i$. $\square$

We will use the following variant. Define for any open set Ω and any compact set ω in $\mathbb{R}^2$

$$r_\Omega(\omega) = \sup_{\substack{K \Subset \Omega \\ \partial K \cap \omega = \varnothing}} r(K \cap \omega). \tag{4.20}$$

In a way, $r_\Omega(\omega)$ counts the radius of the set obtained from ω by discarding the connected components which intersect $\partial\Omega$.

Proposition 4.5. *Assume Ω is open and ω is compact in $\mathbb{R}^2$. Then $2r_\Omega(\omega) \le \mathcal{H}^1(\partial\omega \cap \Omega)$.*

Proof. Let K be a compact subset of Ω such that $\partial K \cap \omega = \varnothing$. Then $\partial(\omega \cap K) = (\partial\omega) \cap K \subset \Omega$ hence from Proposition 4.4,

$$2r(K \cap \omega) \le \mathcal{H}^1(\partial\omega \cap \Omega).$$

The result follows by taking the supremum over K. □

Finally we have:

Proposition 4.6. *Assume ω_1, ω_2 are compact subsets of $\mathbb{R}^2$. Then $r(\omega_1 \cup \omega_2) \le r(\omega_1) + r(\omega_2)$.*

Proof. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite coverings of ω_1 and ω_2 respectively by closed balls, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a covering of $\omega_1 \cup \omega_2$. □

4.4.2 Lower Bound on Initial Balls

Proposition 4.7. *For any $\alpha \in (0,1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $F_\varepsilon(|u|, \Omega) \le \varepsilon^{\alpha-1}$, the following holds.*

There exists a finite collection $\mathcal{B}_0 = \{B_i\}_{i\in I}$ of disjoint closed balls such that, letting $V_0 = \Omega_\varepsilon \cap \cup_{B\in\mathcal{B}_0} B$ we have

1. *$r(\mathcal{B}_0) \le C\varepsilon^{\alpha/2}$, where C is a universal constant.*
2. *$\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \ge \delta\} \subset V_0$, where $\delta = \varepsilon^{\alpha/4}$.*
3. *Let $v = u/|u|$ and for any $t \in (0, 1-\delta)$ let $\omega_t = \{x \in \Omega_\varepsilon \mid |u(x)| \le t\}$. Then*

$$\frac{1}{2}\int_{V_0\setminus\omega_t} |\nabla_A v|^2 + \frac{r(\mathcal{B}_0)^2}{2}\int_{V_0} (\operatorname{curl} A)^2 \ge \pi D_0 \left(\log \frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C\right). \tag{4.21}$$

Here $D_0 = \sum_i |d_i|$, where $d_i = \deg(v, \partial B_i)$ if $B_i \subset \Omega_\varepsilon$ and $d_i = 0$ otherwise, C is a universal constant.

The above result would be true and simpler to prove if we replaced the quantity $r_{\Omega_\varepsilon}(\omega_t)$ in the above result by $r(\omega_t)$, which is larger. However $r(\omega_t)$ cannot be compared to $\mathcal{H}^1(\partial\omega_t \cap \Omega)$, which is what we need in the proof of Theorem 4.1.

4.4.3 Proof of Theorem 4.1

Let $\mathcal{B}_0$ be given by Proposition 4.7. Applying Theorem 4.2 we get a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ of collections of disjoint closed balls. Let $1 > r > r(\mathcal{B}_0)$ and $\mathcal{B} = \mathcal{B}(s)$, with s such that $r(\mathcal{B}) = r$ or equivalently $r = e^s r(\mathcal{B}_0)$. Then items 1 and 2 of Theorem 4.1 follow directly from the corresponding items in Proposition 4.7. It is also obvious that if $\mathcal{B}_1$, $\mathcal{B}_2$ are the collections of balls corresponding to r_1, r_2 with $r_1 > r_2$, then every ball in $\mathcal{B}_2$ is included in one of the balls in $\mathcal{B}_1$. This follows from item 1 in Theorem 4.2.

We turn to the proof of (4.3).

Since from item 2) the map u does not vanish in $\Omega_\varepsilon \setminus V_0$, we may apply Proposition 4.3 in Ω_ε to $v = u/|u|$ and A, to find for every $B \in \mathcal{B}$,

$$E_A\left(v, (B\cap\Omega)\setminus V_0\right) + r(r - r(\mathcal{B}_0))H(A, B\cap\Omega) \geq \pi|d_B|\log\frac{r}{2r(\mathcal{B}_0)},$$

where $d_B = \deg(v, \partial B)$ if $B \subset \Omega_\varepsilon$ and $d_B = 0$ otherwise. Note that if we let $D = \sum_{B\in\mathcal{B}} |d_B|$, then $D \leq D_0$ from Lemma 4.2. Summing this lower bound over all the balls in $\mathcal{B}$ and adding this to (4.21) yields for any $t \in (0, 1-\delta)$

$$\frac{1}{2}\int_{V_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_V (\operatorname{curl} A)^2 \geq \pi D\left(\log\frac{r}{r_{\Omega_\varepsilon}(\omega_t)} - C\right), \tag{4.22}$$

where

$$V = \Omega_\varepsilon \cap \cup_{B\in\mathcal{B}} B, \quad V_t = V\setminus\omega_t = \{x\in V \mid |u(x)| > t\}.$$

The rest of the proof of (4.3) consists in integrating (4.22) with respect to t. It relies heavily on the coarea formula. Let U be the interior of V. Integrating on U or V is equivalent but the coarea formula is best formulated on an open set. As above we let $v = u/|u|$ and

$$U_t = U\backslash\omega_t = \{x \in U, |u| > t\},\ \gamma_t = \{x \in U, |u| = t\},\ \Theta(t) = \frac{1}{2}\int_{U_t} |\nabla_A v|^2.$$

From the Cauchy–Schwarz inequality, we have

$$|\nabla|u||^2 + \frac{(1-t^2)^2}{2\varepsilon^2} \geq |\nabla|u||\frac{\sqrt{2}|1-t^2|}{\varepsilon}.$$

Therefore, using the coarea formula,

$$\frac{1}{2}\int_U |\nabla|u||^2 + \frac{1}{4\varepsilon^2}\int_U \left(1-|u|^2\right)^2 \geq \frac{1}{2}\int_0^{+\infty} \frac{\sqrt{2}|1-t^2|}{\varepsilon}\mathcal{H}^1(\gamma_t)\,dt. \quad (4.23)$$

Also, from Fubini's theorem,

$$\frac{1}{2}\int_U |u|^2|\nabla_A v|^2 = \int_0^{+\infty} -t^2\Theta'(t)\,dt,$$

which yields, after integration by parts,

$$\frac{1}{2}\int_U |u|^2|\nabla_A v|^2 \geq \int_0^{+\infty} 2t\Theta(t)\,dt. \quad (4.24)$$

Let

$$I = \frac{1}{2}\int_U |\nabla_A u|^2 + \frac{1}{4\varepsilon^2}\int_U \left(1-|u|^2\right)^2 + \frac{r^2}{2}\int_U (\operatorname{curl} A)^2.$$

Summing (4.23), (4.24) and since, in view of Lemma 3.4, $|\nabla_A u|^2 = |u|^2|\nabla_A v|^2 + |\nabla|u||^2$, we have

$$I \geq \int_0^1 2t\left[\Theta(t) + \frac{r^2}{2}\int_U (\operatorname{curl} A)^2\right] + \frac{(1-t^2)}{\sqrt{2}\varepsilon}\mathcal{H}^1(\gamma_t)\,dt. \quad (4.25)$$

For any $t \in (0,\delta)$ we claim that $\mathcal{H}^1(\gamma_t) \geq 2r_{\Omega_\varepsilon}(\omega_t)$. Indeed, from Proposition 4.5 we have $2r_{\Omega_\varepsilon}(\omega_t) \leq \mathcal{H}^1(\Omega_\varepsilon \cap \partial\omega_t)$. But $\Omega_\varepsilon \cap \partial\omega_t$ is included in the set of $x \in \Omega_\varepsilon$ such that $|u(x)| = t$. In turn, from Proposition 4.7 and the hypothesis $t \in (0, \delta)$, this set is included in U. It follows that $\Omega_\varepsilon \cap \partial\omega_t \subset \gamma_t$ and then

$$2r_{\Omega_\varepsilon}(\omega_t) \leq \mathcal{H}^1(\gamma_t). \tag{4.26}$$

Now, from (4.22),

$$\Theta(t) + \frac{r^2}{2} \int_U (\operatorname{curl} A)^2 \geq \pi D \left(\log \frac{r}{r_{\Omega_\varepsilon}(\omega_t)} - C \right).$$

Inserting the above and (4.26) into (4.25), we find

$$I \geq \int_0^{1-\delta} 2t\pi D \left(\log \frac{r}{r_{\Omega_\varepsilon}(\omega_t)} - C \right) + \frac{\sqrt{2}(1-t^2)}{\varepsilon} r_{\Omega_\varepsilon}(\omega_t)\, dt.$$

For each t, let us minimize the integrand with respect to $r_{\Omega_\varepsilon}(\omega_t)$. The minimum is achieved for

$$r_{\Omega_\varepsilon}(\omega_t) = \frac{2t\pi\varepsilon D}{\sqrt{2}(1-t^2)}$$

which gives

$$I \geq \int_0^{1-\delta} 2t\pi D \left(\log \frac{r}{\varepsilon D} + f(t) \right) dt,$$

where $f(t) = \log(\frac{1-t^2}{\sqrt{2}\pi t}) - C$. Therefore

$$I \geq \pi D \left((1-\delta)^2 \log \frac{r}{\varepsilon D} - C \right),$$

where C is a universal constant, namely the integral of the function $t \to -2tf(t)$ on $[0,1]$.

If $\pi D \left(\log \frac{r}{\varepsilon D} - C \right) \leq 0$, then the relation (4.3) is trivially true. If not, then we can write

$$I \geq \pi D \left(\log \frac{r}{\varepsilon D} - 2\delta \log \frac{r}{\varepsilon D} - C \right).$$

Since $r \leq 1$ and $D \geq 1$ (the case $D = 0$ was excluded), the contribution of $-2\delta \log(r/D)$ to the right-hand side is positive. On the other hand,

$\delta = \varepsilon^{\alpha/4}$ therefore, if ε is small enough depending on α, then $\delta|\log\varepsilon| \le 1$ and

$$I \ge \pi D \left(\log \frac{r}{\varepsilon D} - 1 - C\right).$$

This proves (4.3); it remains to prove (4.4).

Let $M = F_\varepsilon(u, A, \Omega)$. To prove (4.4), we may use (4.23) together with a mean value argument to find a regular value $t \in (1/2, 3/4)$ of $|u|$ such that $2r_{\Omega_\varepsilon}(\omega_t) \le \mathcal{H}^1(\gamma_t) \le C\varepsilon M$, where C is a universal constant. Applying (4.26) and Proposition 4.7 we find

$$\frac{1}{2}\int_{U_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_U (\operatorname{curl} A)^2 \ge \pi D \left(\log \frac{r}{C\varepsilon M} - C\right).$$

but since $t \in (1/2, 3/4)$ and $|u| > t$ on U_t it holds that $|\nabla_A v|^2 \le 4|\nabla_A u|^2$ on U_t therefore

$$4M \ge \frac{1}{2}\int_{U_t} |\nabla_A v|^2 + \frac{r^2}{2}\int_U (\operatorname{curl} A)^2 \ge \pi D \left(\log \frac{r}{C\varepsilon M} - C\right). \tag{4.27}$$

We conclude by noting that $r/\varepsilon M \ge \varepsilon^{-\alpha/2}$ which together with (4.27) implies

$$CM \ge D(\alpha|\log\varepsilon| - 1)$$

hereby proving (4.4) if ε is small enough depending on α.

The rest of the chapter is devoted to the proof of Proposition 4.7.

4.5 Proof of Proposition 4.7

4.5.1 Initial Set

Proposition 4.8. *For any $M, \varepsilon, \delta > 0$ satisfying $\varepsilon, \delta < 1$, any $u \in C^1(\Omega, \mathbb{C})$ satisfying $F_\varepsilon(|u|, \Omega) \le M$, we have*

$$r\left(\{x \in \Omega_\varepsilon, ||u(x)| - 1| \ge \delta\}\right) \le C\frac{\varepsilon M}{\delta^2},$$

for some universal constant C, where $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Proof. Let $\rho = |u|$. Then

$$\frac{1}{2}\int_\Omega |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho^2)^2 \le M$$

and thus using the Cauchy–Schwarz inequality as before

$$\int_\Omega |\nabla\rho|\frac{|1-\rho^2|}{\sqrt{2}\varepsilon} \le M.$$

Using the coarea formula, we find

$$\int_{t\in\mathbb{R}} \frac{|1-t^2|}{\varepsilon}\mathcal{H}^1\left(\{x\in\Omega, \rho(x)=t\}\right)\,dt \le \sqrt{2}M. \tag{4.28}$$

Then from (4.28) and the mean value theorem there exists $t\in(1-\delta,1-\delta/2)$ such that

$$\mathcal{H}^1\left(\{x\in\Omega \mid |1-\rho(x)|=t\}\right) \le 2\sqrt{2}\frac{M\varepsilon}{\delta|1-t^2|} \le 4\sqrt{2}\frac{M\varepsilon}{\delta^2},$$

indeed $|1-t^2|\ge\delta/2$ if $t\in(1-\delta,1-\delta/2)$.

Letting $\omega=\{|\rho-1|\ge t\}$, we have

$$\mathcal{H}^1(\partial\omega\cap\Omega)\le C\frac{M\varepsilon}{\delta^2}. \tag{4.29}$$

It follows from

$$\frac{1}{4\varepsilon^2}\int_\Omega (1-\rho^2)^2 \le M$$

and $(1-t^2)^2\ge\delta^2/4$ that

$$|\omega| \le \frac{16M\varepsilon^2}{\delta^2}.$$

Therefore there exists some $s\in(0,\varepsilon)$ such that the length of $\gamma=\{x\in\omega \mid \mathrm{dist}(x,\partial\Omega)=s\}$ is less than $16\varepsilon M/\delta^2$. Letting $\Omega_s=\{x\in\Omega \mid \mathrm{dist}(x,\partial\Omega)>s\}$, we have $\omega\cap\partial\Omega_s\subset\gamma$ hence

$$\mathcal{H}^1(\omega\cap\partial\Omega_s)\le C\frac{M\varepsilon}{\delta^2}. \tag{4.30}$$

Then (4.29) and (4.30) yield $\mathcal{H}^1(\partial(\omega\cap\Omega_s))\le CM\varepsilon/\delta^2$. Hence from Proposition 4.4,

$$r(\omega\cap\Omega_s)\le C\frac{M\varepsilon}{\delta^2}.$$

Since $\{x\in\Omega_\varepsilon \mid |\rho(x)-1|\ge 1-\delta\}\subset\omega\cap\Omega_s$, the proposition is proved.

□

4.5.2 Construction of the Appropriate Initial Collection

In this proof C denotes a generic universal constant. By a *collection* of balls, we will always mean a finite collection of disjoint closed balls.

From Proposition 4.8 applied to u with $M = \varepsilon^{\alpha-1}$ and $\delta = \varepsilon^{\alpha/4}$ the set $\{x \in \Omega_\varepsilon \mid ||u(x)| - 1| \geq \delta\}$ has radius less than $C\varepsilon^{\alpha/2}$ and thus may be covered by a union of disjoint closed balls U such that

$$R := r(U) \leq C\varepsilon^{\alpha/2}. \tag{4.31}$$

The difficulty here consists in finding a collection of balls which works for each t and contains enough energy, so we split the energy we wish to bound from below as the energy on $\mathcal{B}_0 \backslash U$ plus the energy over $U \backslash \omega_t$. We choose a set K in order to maximize the first contribution (this is independent of t). We add to it U and balls obtained by growing the ω_t with smallest total degree, and we finally cover the whole set by balls which are the desired $\mathcal{B}_0$. Let us now go into details.

— *Step 1:* We may write U as a disjoint union $U_0 \cup U_1$, where U_0 contains those balls in U which intersect $\partial\Omega_\varepsilon$, and U_1 contains the remaining balls. Then we define (see Fig. 4.2)

$$\tilde{\Omega} = \Omega_\varepsilon \setminus U_0.$$

Now for any $t \in (0, 1-\delta)$, we claim that

$$r_{\Omega_\varepsilon}(\omega_t) \geq r(\omega_t \cap \tilde{\Omega}). \tag{4.32}$$

Indeed, since ω_t is contained in the interior of U, the set $\omega_t \cap \tilde{\Omega}$ is contained in the interior of U_1, which is a compact subset of Ω_ε. Thus $r(\omega_t \cap \tilde{\Omega}) = r(\omega_t \cap U_1)$ and from (4.20) we have $r_{\Omega_\varepsilon}(\omega_t) \geq r(\omega_t \cap U_1)$. The inequality (4.32) follows.

— *Step 2:* For every $t \in (0, 1-\delta)$, the set $\omega_t \cap \tilde{\Omega}$ may be covered by a collection of balls $\mathcal{B}_t^0$ of total radius no greater than $2r(\omega_t \cap \tilde{\Omega})$. Then using Theorem 4.2 and since $r(\omega_t \cap \tilde{\Omega}) \leq R$, these balls may be grown into a collection $\mathcal{B}_t$ such that

$$r(\mathcal{B}_t) = 2R \tag{4.33}$$

and, from Proposition 4.3 applied in $\tilde{\Omega}$ to $v = u/|u|$ and A, we have summing (4.15) over all balls in $\mathcal{B}_t$,

$$E_A(v, V_t \setminus \omega_t) + 4R^2 H(A, V_t) \geq \pi D_t \left(\log \frac{2R}{2r(\omega_t \cap \tilde{\Omega})} - \log 2 \right), \tag{4.34}$$

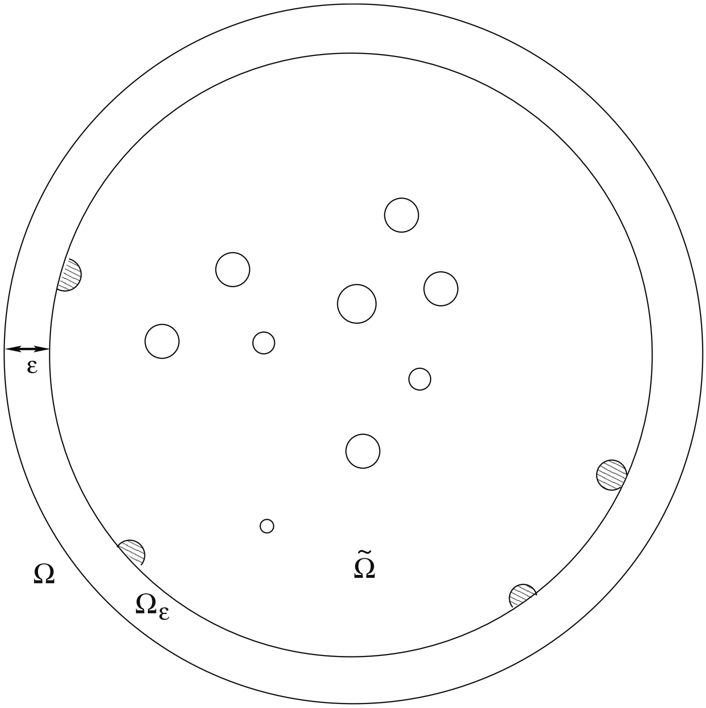

Figure 4.2: $\tilde{\Omega}$

where we have used the notation (4.7), (4.8), where V_t is the union of balls in $\mathcal{B}_t$ intersected with $\tilde{\Omega}$ and where $D_t = \sum_B |\deg(u, \partial B)|$, the sum running over the balls $B \in \mathcal{B}_t$ which are included in $\tilde{\Omega}$.

There exists $\bar{t} \in (0, 1-\delta)$ such that $D_{\bar{t}}$ is minimal. We let

$$\mathcal{B} = \mathcal{B}_{\bar{t}}. \tag{4.35}$$

— *Step 3:* Letting m denote the supremum of

$$\mathcal{F}(K) = E_A\left(v, (K \cap \tilde{\Omega}) \setminus U\right) + 4R^2 H(A, K \cap \tilde{\Omega}),$$

where the supremum runs over compact sets $K \Subset \Omega$ such that $r(K) < 2R$; we can find such a K such that $r(K) < 2R$ and $\mathcal{F}(K) \geq m - 1$. Note, in particular, that from (4.33) we have

$$\mathcal{F}(K) + 1 \geq \mathcal{F}(V_t), \quad \text{for every } t \in (0, 1-\delta).$$

We define $\mathcal{B}_0$ to be a collection of disjoint closed balls which cover the balls in $\mathcal{B}$ defined by (4.35), as well as K and U. There exists such a collection with total radius $5R$. Clearly, from (4.31), item 1 of Proposition 4.7 is satisfied and item 2 is satisfied as well. It remains to check (4.21).

— *Step 4:* Let K_0 be the union of balls in $\mathcal{B}_0$ intersected with Ω_ε. We have

$$I := E_A(v, K_0 \setminus \omega_t) + r(\mathcal{B}_0)^2 H(A, K_0) \geq \mathcal{F}(K) + E_A(v, U \setminus \omega_t).$$

It follows, by the definition of K and (4.34), that for every $t \in (0, 1-\delta)$

$$\begin{aligned} I + 1 \geq \mathcal{F}(V_t) + E_A(v, U \setminus \omega_t) &\geq E_A(v, V_t \setminus \omega_t) + 4R^2 H(A, V_t) \\ &\geq \pi D_t \log \frac{2R}{4r(\omega_t \cap \tilde{\Omega})}. \end{aligned}$$

From (4.32) and since $r(\mathcal{B}_0) = 5R$, the right-hand side is larger than

$$\pi D_t \left(\log \frac{r(\mathcal{B}_0)}{r_{\Omega_\varepsilon}(\omega_t)} - C \right),$$

thus (4.21) will be satisfied if we prove that $D_t \geq D_0$ for every t. By definition of $\mathcal{B}$, we have $D_t \geq D_{\tilde{\Omega}}(\mathcal{B})$, where we have used the notation $D_{\tilde{\Omega}}(\mathcal{B})$ for the sum $\sum_B |\deg(u, \partial B)|$, where the sum runs over the balls in $\mathcal{B}$ which are included in $\tilde{\Omega}$. But since $\mathcal{B}_0$ covers the balls in $\mathcal{B}$, we have from Lemma 4.2 that $D_{\tilde{\Omega}}(\mathcal{B}) \geq D_{\tilde{\Omega}}(\mathcal{B}_0)$ and therefore $D_t \geq D_{\tilde{\Omega}}(\mathcal{B}_0)$. It remains to remark that

$$D_0 := D_{\Omega_\varepsilon}(\mathcal{B}_0) = D_{\tilde{\Omega}}(\mathcal{B}_0).$$

Indeed if $B \in \mathcal{B}_0$ and $B \subset \Omega_\varepsilon$, then the balls of U which are included in B are included in Ω_ε, hence are in U_1. Therefore $B \cap U_0 = \varnothing$ and $B \subset \tilde{\Omega}$.

Bibliographic notes on Chapter 4: As we mentioned, the material presented here is an improvement of the results of the sequence [166, 169, 170, 175], using the ball-growth idea first introduced independently by Jerrard and Sandier in [113, 166]. The construction of [113] yields results in n dimensions for the corresponding n-energy.

Several ball-constructions were previously introduced in the literature, all dealing with numbers of vortices bounded independently of ε.

First, in Bethuel–Brezis–Hélein [43], the balls are defined as a disjoint covering of $|u| \leq \frac{1}{2}$ by balls of radius less than $C\varepsilon$, in number bounded independently of ε. This requires u to be a solution of (1.3) and the upper bound on the energy $E_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$ to hold. In Bethuel–Rivière [52] appears the idea of using lower bounds on annuli of larger size than the balls of [43]. The method again crucially uses the energy upper bound, and the equation through the Pohozaev identity. It yields balls of radii ε^α, $\alpha < 1$ with corresponding energy lower bounds. This method was later extended to nonsolutions by Almeida–Bethuel [14] via a parabolic regularisation of the map u.

Chapter 5

Coupling the Ball Construction to the Pohozaev Identity and Applications

The key ingredient here is the Pohozaev identity for solutions of Ginzburg–Landau. This identity was already used crucially in Bethuel–Brezis–Hélein [43], Brezis–Merle–Rivière [61], and its first use on small balls goes back to Bethuel–Rivière [52] and Struwe [189]. Its consequences were also explored further in the book of Pacard–Rivière [148]. Here, the idea is to combine it with the ball-construction method in order to obtain lower bounds for the energy in terms of the potential term $\int(1-|u|^2)^2$ instead of the degree, or equivalently, upper bounds of the potential by the energy divided by $|\log \varepsilon|$. This method works for solutions of the Ginzburg–Landau equation, without magnetic field as well as with. We will present the two situations in parallel, in Sections 5.1 and 5.2. In the third section of the chapter, we present applications to the microscopic analysis of vortices of solutions of (GL) or (1.3). Among all these results, only Theorem 5.4 will be used later, for the study of solutions with bounded numbers of vortices: for Proposition 10.2 and in the course of the proof of Theorem 11.1.

5.1 The Case of Ginzburg–Landau without Magnetic Field

For simplicity we start with the case of solutions of Ginzburg–Landau without magnetic field, i.e., we consider u which satisfies

$$-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \Omega \tag{5.1}$$

with either Dirichlet boundary condition $u = g$ on $\partial\Omega$, $|g| = 1$, and Ω starshaped, or Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. We recall the Ginzburg–Landau energy without magnetic field is written

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

The Pohozaev identity consists in multiplying the equation (5.1) by $x \cdot \nabla u$ where x is the coordinate centered at some point, and integrating over the ball of radius r. If $B(x_0, r) \cap \partial\Omega = \varnothing$, it gives

$$\frac{1}{r}\int_{B(x_0,r)} \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{\partial B(x_0,r)} \left(\left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 \right) + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{5.2}$$

Integrating this relation over r will yield bounds on the energy on annuli in terms of $\int \frac{(1-|u|^2)^2}{\varepsilon^2}$. The main difficulty is to deal with the case of balls intersecting $\partial\Omega$. To handle this, we will perform a reflection in the Neumann case.

Theorem 5.1 (Pohozaev ball construction). *Let u be a solution of (5.1). Let $\mathcal{B}_0$ be a finite collection of disjoint closed balls and let $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ satisfy the results of Theorem 4.2. Then, letting $r_0 = r(\mathcal{B}_0)$, there exists a constant $C(\Omega)$ such that $\forall r_0 < r_1 < C(\Omega)$, and s being such that $r(\mathcal{B}(s)) = r_1$, we have*

1. *For every $B \in \mathcal{B}(s)$ such that $B \subset \Omega$,*

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq \left(\frac{1}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2}\right) \log\frac{r_1}{r_0} \tag{5.3}$$

2. *If Ω is strictly starshaped and u satisfies the fixed Dirichlet boundary condition $u = g$, $|g| = 1$, for every $B \in \mathcal{B}(s)$ intersecting $\partial\Omega$,*

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq \left(\frac{1}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2}\right) \log\frac{r_1}{r_0} - Cr_1, \tag{5.4}$$

   where C depends on Ω and g.

3. *If u satisfies the Neumann boundary condition, there exists a finite collection of disjoint closed balls $\mathcal{B}'$ covering $\cup_{B\in\mathcal{B}(s)} B$ such that $r(\mathcal{B}') \leq Cr_1$, and for every $B \in \mathcal{B}'$ such that $B \subset \Omega$, (5.3) holds, while for every $B \in \mathcal{B}'$ intersecting $\partial\Omega$,*

$$C \int_B |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq \left(\frac{1}{2} \int_{\mathcal{B}_0 \cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} \right) \log \frac{r_1}{r_0}, \qquad (5.5)$$

where C depends on Ω.

Remark 5.1. 1. Since the balls are disjoint, these estimates can be summed over all the balls to give a single estimate over the union.

2. We do not need any assumption of the energy of u, rather this proves a lower bound for it.

We start with a lemma, which is a generalization of (5.2). Let us denote by

$$T_{ij} = (\partial_i u, \partial_j u) - \frac{1}{2}\left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right) \delta_{ij}, \qquad (5.6)$$

the "stress-energy tensor" as in Definition 3.4, but without magnetic field. As in Proposition 3.7, a direct calculation yields that

$$\partial_1 T_{1i} + \partial_2 T_{2i} = \left(\partial_i u, \left(\Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \right) \right)$$

hence if u is a solution of (5.1), we have

$$\partial_1 T_{1i} + \partial_2 T_{2i} = 0 \quad \text{for every} \quad i = 1, 2. \qquad (5.7)$$

Lemma 5.1. *Let u be a solution of (5.1) and U be an open subset of Ω, then, for every vector-field X,*

$$\int_{\partial U} \sum_{i,j} X_j \nu_i T_{ij} = \int_U \sum_{i,j} (\partial_i X_j) T_{ij}, \qquad (5.8)$$

where the indices i, j run over $1, 2$ and ν denotes the outer unit normal to ∂U.

Proof. This relation comes from multiplying (5.7) by X_i, summing over i and integrating over U. In short notation, it yields $\int_U X_1 \operatorname{div} T_{i1} + X_2 \operatorname{div} T_{i2} = 0$. Integrating by parts leads to (5.8). ☐

Observe that, τ denoting the unit tangent vector to ∂U, we have on ∂U

$$\sum_{i,j} X_j \nu_i T_{ij} = X_\nu T_{\nu\nu} + X_\tau T_{\nu\tau} \tag{5.9}$$

in obvious notation (with $X_\nu = X \cdot \nu$ and $X_\tau = X \cdot \tau$).

The Pohozaev identity (5.2) follows by taking the particular choice $X(x) = x - x_0$ and $U = B(x_0, r)$. It was used in [43] to obtain the following result.

Lemma 5.2 (Boundedness of the potential [43]**).** *Assume Ω is strictly starshaped and u is a solution of (5.1) with $u = g$ on $\partial\Omega$ and $|g| = 1$ (g independent of ε), then there exists a constant C depending only on g and Ω such that*

$$\int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} + \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \le C. \tag{5.10}$$

Proof. Assume that Ω is strictly starshaped around x_0 and apply Lemma 5.1 in Ω with the particular choice $X(x) = x - x_0$. We then have $\partial_i X_j = \delta_{ij}$, and hence

$$\sum_{i,j} (\partial_i X_j) T_{ij} = T_{11} + T_{22} = -\frac{1}{2\varepsilon^2}(1-|u|^2)^2. \tag{5.11}$$

Also on $\partial\Omega$, $X_\nu T_{\nu\nu} + X_\tau T_{\nu\tau}$ is equal to

$$\frac{1}{2}\left(|\partial_\nu u|^2 - |\partial_\tau u|^2 - \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)(x-x_0)\cdot\nu + (\partial_\tau u, \partial_\nu u)(x-x_0)\cdot\tau.$$

Combining this with (5.8) and (5.9), we are led to

$$\int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{\partial\Omega} (x-x_0)\cdot\nu \left(\left|\tfrac{\partial g}{\partial \tau}\right|^2 - \left|\tfrac{\partial u}{\partial \nu}\right|^2\right) - 2(x-x_0)\cdot\tau\left(\tfrac{\partial g}{\partial\tau}, \tfrac{\partial u}{\partial \nu}\right). \tag{5.12}$$

Since Ω is strictly starshaped around x_0, there exists a constant $\alpha > 0$ such that $(x-x_0)\cdot\nu > \alpha$ on $\partial\Omega$. Using this and a Cauchy Schwarz inequality in (5.12), we find

$$\int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} \le C\int_{\partial\Omega}\left|\frac{\partial g}{\partial\tau}\right|^2 - \alpha\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 - C\int_{\partial\Omega}\left(\frac{\partial g}{\partial\tau},\frac{\partial u}{\partial\nu}\right)$$
$$\le C\int_{\partial\Omega}\left|\frac{\partial g}{\partial\tau}\right|^2 - \alpha\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{\alpha}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 + C\int_{\partial\Omega}\left|\frac{\partial g}{\partial\tau}\right|^2$$

where the constant C depends only on Ω. Consequently,

$$\int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} + \frac{\alpha}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 \le C$$

where C depends on Ω and g, and thus the lemma is proved. □

Proof of Theorem 5.1, interior result and Dirichlet case. The method consists in a ball-growth procedure, as in Chapter 4.

Following Chapter 4, let us denote by

$$\mathcal{F}(x,r) = \frac{1}{2}\int_{B(x,r)\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

It is easy to check that $\mathcal{F}$ is monotonic (in the sense of Definition 4.1).

Given the family $\{\mathcal{B}(t)\}$, $s>0$ and $r_1 = r(\mathcal{B}(s))$, applying Proposition 4.1, we obtain that for every $B\in\mathcal{B}(s)$,

$$\mathcal{F}(B) - \mathcal{F}(B\cap\mathcal{B}_0) \ge \int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} r\frac{\partial\mathcal{F}}{\partial r}(x,r)\,dt. \tag{5.13}$$

But,

$$r\frac{\partial\mathcal{F}}{\partial r}(x,r) = \frac{r}{2}\int_{\partial B(x,r)\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{5.14}$$

Now let x_0 be any point in Ω and let us apply Lemma 5.1 in $B(x_0,r)$

with $X(x) = x - x_0$. Using (5.11) and (5.9) as before, we find the relation

$$\int_{B(x_0,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{\partial(B(x_0,r)\cap\Omega)} (x-x_0)\cdot\nu\left(\left|\frac{\partial u}{\partial\tau}\right|^2 - \left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right) - 2(x-x_0)\cdot\tau\left(\frac{\partial u}{\partial\nu}, \frac{\partial u}{\partial\tau}\right). \quad (5.15)$$

Observe that (5.2) follows if $B(x_0, r)\cap\partial\Omega = \varnothing$. Therefore, if $B \in \mathcal{B}(s)$ is such that $B \subset \Omega$, it does not intersect $\partial\Omega$ and, combining (5.2) and (5.14), we can write that

$$r\frac{\partial\mathcal{F}}{\partial r}(x,r) = \frac{1}{2}\int_{B(x,r)} \frac{(1-|u|^2)^2}{\varepsilon^2} + r\int_{\partial B(x,r)} \left|\frac{\partial u}{\partial\nu}\right|^2 \geq \frac{1}{2}\int_{B(x,r)} \frac{(1-|u|^2)^2}{\varepsilon^2}. \quad (5.16)$$

Inserting this into (5.13), and using the fact that $\mathcal{B}(t)$ always contains $\mathcal{B}_0 \cap B$, we are led to

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq \int_0^s \frac{1}{2}\int_{\mathcal{B}(t)\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2}\,dt \geq \frac{s}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} = \frac{1}{2}\log\frac{r_1}{r_0}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2}, \quad (5.17)$$

in view of the definition of r_1. This concludes the proof of item 1).

Let us now prove item 2). In the Dirichlet case, let us return to (5.15). Since $\frac{\partial u}{\partial\tau}$ then depends only on g, $|u| = 1$ on $\partial\Omega$, and since (5.10) holds, we see that the contributions on $\partial\Omega\cap B(x_0, r)$ are $O(|x - x_0|) = O(r)$

as $r \to 0$ and thus

$$\int_{B(x,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} = r \int_{\partial B(x,r)\cap\Omega} \left(\left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 \right) + \frac{(1-|u|^2)^2}{2\varepsilon^2} + O(r)$$

In the case where B intersects $\partial\Omega$, we can write in place of (5.16),

$$\begin{aligned} r\frac{\partial \mathcal{F}}{\partial r}(x,r) &= \frac{1}{2} \int_{B(x,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} + r \int_{\partial B(x,r)\cap\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 + O(r) \\ &\geq \frac{1}{2} \int_{B(x,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} + O(r). \end{aligned}$$

Then, in place of (5.17),

$$\begin{aligned} \frac{1}{2} \int_{B\backslash\mathcal{B}_0} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} &\geq \int_0^s \frac{1}{2} \int_{\mathcal{B}(t)\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} - O(r(\mathcal{B}(t))\,dt \\ &\geq \frac{s}{2} \int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} - O\left(\int_0^s (e^t r(\mathcal{B}_0))\,dt \right) \\ &= \frac{1}{2} \int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} \log\frac{r_1}{r_0} - O(r(\mathcal{B}(s))). \end{aligned}$$

And since $r(\mathcal{B}(s)) = r_1$, we conclude that (5.4) holds. □

Proof of Theorem 5.1 in the Neumann case. In this case, we need to extend u to a slightly larger domain $\tilde{\Omega}$ through a reflection. Thus let $\tilde{\Omega}$ denote the tubular neighborhood of size R of Ω, i.e., $\Omega \subset \tilde{\Omega}$. The procedure is as follows: let ψ be a smooth mapping of Ω onto the unit disc. It can be extended to a mapping from $\tilde{\Omega}$ to a domain strictly containing the unit disc. Then let $\mathcal{R}$ denote the reflection with respect to the unit circle defined in complex coordinates by $\mathcal{R}(z) = \frac{z}{|z|^2}$. The mapping $\varphi = \psi^{-1} \circ \mathcal{R} \circ \psi$ then maps $\tilde{\Omega}\backslash\Omega$ to Ω. One can check that it is the identity on $\partial\Omega$, that it is C^2 in $\tilde{\Omega}\backslash\Omega$, and that $D\varphi(x)$ converges to the orthogonal reflection relative to the tangent to $\partial\Omega$ at x_0 as $x \to x_0 \in \partial\Omega$, at a rate bounded by $C|x - x_0|$.

We can then extend u, the solution of (5.1) with $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, by $\overline{u} = u$ in Ω and

$$\overline{u}(x) = u(\varphi(x)) \quad \text{if } x \in \tilde{\Omega}\backslash\Omega.$$

Since $D\varphi$ converges to a reflection with respect to the boundary as $x \to \partial\Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, we find that $\overline{u}$ is C^1 in $\tilde{\Omega}$.

The method is the same as before, i.e., we define

$$\mathcal{F}(x,r) = \frac{1}{2} \int\limits_{B(x,r)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}. \tag{5.18}$$

Again $\mathcal{F}$ is monotonic and

$$r\frac{\partial \mathcal{F}}{\partial r}(x,r) = \frac{r}{2} \int\limits_{\partial B(x,r)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}.$$

Again, given $\{\mathcal{B}(t)\}$, we consider s such that $r_1 = r(\mathcal{B}(s)) < R$. We need to add to the collection of balls $\mathcal{B}(s)$ the $\varphi(B \cap (\tilde{\Omega}\backslash\Omega))$ for all $B \in \mathcal{B}(s)$ which intersect $\partial\Omega$. The $\varphi(B \cap (\tilde{\Omega}\backslash\Omega))$ are not balls, however their total radius is controlled by Cr_1, and thus they can be covered by a finite collection of disjoint closed balls of total radius $\leq Cr_1$. Let us add them to the collection $\mathcal{B}(s)$. These new balls may intersect some of the balls in $\mathcal{B}(s)$. If this is the case, then we perform merging of the intersecting balls according to Lemma 4.1, until we obtain a family of disjoint closed balls, still of radius $\leq Cr_1$. This is the final family we need, it is denoted $\mathcal{B}'$. Observe that by construction, any ball in $\mathcal{B}'$ that did not belong to the collection $\mathcal{B}(s)$ has to intersect $\partial\Omega$. Hence, all the balls $B \in \mathcal{B}'$ which do not intersect $\partial\Omega$ are balls of $\mathcal{B}(s)$ and for them, the proof of item 1) applies and gives the result.

We now only need to deal with the final balls which intersect $\partial\Omega$. Since $r(\mathcal{B}') \leq Cr_1$, they will always remain inside $\tilde{\Omega}$ if $r_1 < R/C$. We claim that

Lemma 5.3. *For every $B(x_0,r) \subset \tilde{\Omega}$, we have*

$$\int\limits_{B(x_0,r)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} + O\left(r\mathcal{F}(x_0,r)\right) =$$

$$r \int\limits_{\partial B(x_0,r)} \left|\frac{\partial \overline{u}}{\partial \tau}\right|^2 - \left|\frac{\partial \overline{u}}{\partial \nu}\right|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2} + O\left(r^2 \frac{\partial \mathcal{F}}{\partial r}(x_0,r)\right). \tag{5.19}$$

The proof is postponed until later in this section.

Now let $B \in \mathcal{B}(s)$ be a ball possibly intersecting $\partial\Omega$. Applying Proposition 4.1 to (5.18), we obtain that

$$\mathcal{F}(B) - \mathcal{F}(B \cap \mathcal{B}_0) \geq \int_0^s \sum_{B(x,r) \in \mathcal{B}(t) \cap B} r \frac{\partial \mathcal{F}}{\partial r}(x,r)\, dt. \tag{5.20}$$

But

$$r \frac{\partial \mathcal{F}}{\partial r}(x,r) = \frac{r}{2} \int_{\partial B(x,r)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}.$$

In view of Lemma 5.3, we can then write that

$$\begin{aligned} r \frac{\partial \mathcal{F}}{\partial r}(x,r) &= \frac{1}{2} \int_{B(x,r)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} + r \int_{\partial B(x,r)} \left| \frac{\partial \overline{u}}{\partial \nu} \right|^2 \\ &\quad + O\left(r^2 \frac{\partial \mathcal{F}}{\partial r}(x,r) \right) + O\left(r \mathcal{F}(x,r) \right) \\ &\geq \frac{1}{2} \int_{B(x,r)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} + O\left(r^2 \frac{\partial \mathcal{F}}{\partial r}(x,r) \right) + O\left(r \mathcal{F}(x,r) \right). \end{aligned}$$

Let us sum these relations over the balls $B(x,r) \in \mathcal{B}(t)$ which are included in B, and integrate this relation for $t \in [0,s]$. After integration, the errors on the right-hand side are bounded respectively by $r(\mathcal{B}(s))\mathcal{F}(B)$ (this follows from (5.20)) and by $\mathcal{F}(B) \int_0^s e^t r(\mathcal{B}(0))\, dt = r(\mathcal{B}(s))\mathcal{F}(B) = r_1 \mathcal{F}(B)$. Finally, inserting this into (5.20), we are led to

$$\begin{aligned} \frac{1}{2} \int_{B \setminus \mathcal{B}_0} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2} &\geq \int_0^s \frac{1}{2} \int_{\mathcal{B}(t) \cap B} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2}\, dt - C r_1 \mathcal{F}(B) \\ &\geq \frac{s}{2} \int_{\mathcal{B}_0 \cap B} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} - C r_1 \mathcal{F}(B). \end{aligned}$$

And since $s = \log \frac{r_1}{r_0}$, we conclude that

$$\left(\frac{1}{2} + C r_1 \right) \int_B |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2} \geq \frac{1}{2} \int_{\mathcal{B}_0 \cap B} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} \log \frac{r_1}{r_0} \tag{5.21}$$

but

$$\int_B |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2} = \frac{1}{2} \int_{B\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \frac{1}{2} \int_{B\cap(\tilde{\Omega}\setminus\Omega)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}.$$

Moreover, doing a change of variables, from the properties of φ,

$$\int_{B\cap(\tilde{\Omega}\setminus\Omega)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2} \leq (1+O(r)) \int_{\varphi(B\cap(\tilde{\Omega}\setminus\Omega))} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

Returning to (5.21) we may write that

$$\int_{\mathcal{B}_0\cap B} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} \log \frac{r_1}{r_0} \leq \tag{5.22}$$
$$C\left(\int_{B\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \int_{\varphi(B\cap(\tilde{\Omega}\setminus\Omega))} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right).$$

Now let B_1 be a ball belonging to the final family $\mathcal{B}'$ and intersecting $\partial\Omega$. We may add up the relations (5.22) obtained for all of the balls $B \in \mathcal{B}(s)$ contained in $\mathcal{B}(s)$ such that $B \subset B_1$ or $\varphi(B) \subset B_1$. Since these B's are disjoint (and so are the $\varphi(B)$'s), each point in B_1 belongs to at most one ball $B \in \mathcal{B}(s)$ and/or one $\varphi(B)$, so is at most counted twice. This means we can write

$$\int_{\mathcal{B}_0\cap(\cup_{B\subset B_1} B)} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} \log \frac{r_1}{r_0} \leq C \int_{B_1} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

Since $\mathcal{B}_0$ is covered by the collection $\mathcal{B}(s)$, we have $\mathcal{B}_0 \cap (\cup_{B\subset B_1} B) = \mathcal{B}_0 \cap B_1$, and we may conclude that (5.5) holds for B_1. □

Proof of Lemma 5.3. If $B(x_0, r)$ does not intersect $\partial\Omega$, then this was already established. Let thus $B(x_0, r)$ be a ball intersecting $\partial\Omega$.

Let $D_1 = B(x_0, r) \cap \Omega$ and $D_2 = B(x_0, r) \cap (\mathbb{R}^2 \setminus \Omega)$. We may apply directly (5.15) in D_1 and get

$$\int_{B(x_0,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{B(x_0,r)\cap\partial\Omega} (x-x_0)\cdot\nu \left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) + \int_{\partial B(x_0,r)\cap\Omega} r \left(\left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) \quad (5.23)$$

where the terms in $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ have vanished due to the Neumann boundary condition.

Let us now write a Pohozaev type identity on $D_2' = \varphi(D_2) \subset \Omega$. We define $y = \varphi(x)$ and $Y(y) = D\varphi(x)(x-x_0)$, i.e., the push-forward of the vector field $X(x) = x - x_0$ by φ. Applying Lemma 5.1 to the vector field Y on D_2', we find

$$\int_{D_2'} \sum_{i,j} (\partial_i Y_j) T_{ij} = \int_{\partial D_2'} \sum_{i,j} Y_j \nu_i T_{ij}. \quad (5.24)$$

Let us study each of the terms in (5.24). First, since $Y_j(y) = \sum_k \partial_k \varphi_j(x) X_k(x) = \sum_k \frac{\partial y_j}{\partial x_k} X_k(x)$, we have

$$\frac{\partial Y_j}{\partial y_i} = \sum_{k,l} \frac{\partial^2 y_j}{\partial x_k \partial x_l} \frac{\partial x_l}{\partial y_i} X_k(x) + \frac{\partial y_j}{\partial x_k} \frac{\partial X_k}{\partial x_l} \frac{\partial x_l}{\partial y_i}.$$

But $\frac{\partial X_k}{\partial x_l} = \delta_{kl}$, hence we find

$$\frac{\partial Y_j}{\partial y_i} = \left(\sum_{k,l} \frac{\partial^2 y_j}{\partial x_k \partial x_l} \frac{\partial x_l}{\partial y_i} X_k(x) \right) + \delta_{ij}.$$

In view of the behavior of $y = \varphi(x)$ mentioned above (bounded second derivative, invertible first differential), the first term on the right-hand side is bounded by a constant times $|X| = |x - x_0| = r$. In other words

$$\partial_i Y_j = \delta_{ij} + O(r).$$

Inserting this into (5.24) and using the expression of T_{ij}, we find that the left-hand side of (5.24) is

$$-\frac{1}{2} \int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2} + O(r|T_{ij}|).$$

Since $|T_{ij}| \leq C\left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)$ this can be written

$$\int_{D_2'} \sum_{i,j} (\partial_i Y_j) T_{ij} = -\frac{1}{2} \int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2} + O(r E_\varepsilon(u, B(x_0, r))), \tag{5.25}$$

where $E_\varepsilon(u, U)$ denotes

$$\frac{1}{2} \int_U |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}.$$

Let us now deal with the right-hand side of (5.24). We recall that

$$\int_{\partial D_2'} \sum_{i,j} Y_j \nu_i T_{ij} = \int_{\partial D_2'} Y_\nu T_{\nu\nu} + Y_\tau T_{\nu\tau}.$$

There are two contributions to this term, the one on $\partial\Omega$ and the one on $\partial D_2' \cap \Omega$. For the term on $\partial\Omega$, observe that from the Neumann boundary condition we have $T_{\nu\tau} = 0$ on $\partial\Omega$. Also $Y_\nu = Y \cdot \nu = -(x - x_0) \cdot \nu$ by definition of $Y = D\varphi(x - x_0)$ (since $D\varphi$ coincides with the reflection with respect to $\partial\Omega$ on $\partial\Omega$). Thus the contribution of that part is

$$\int_{\partial\Omega \cap B(x_0, r)} (x - x_0) \cdot \nu \left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right).$$

For the contribution on $\partial D_2' \cap \Omega$, we use the fact that $D\varphi$ is the reflection relative to the tangent to $\partial\Omega$ up to $O(r)$, hence $Y_\nu = Y \cdot \nu = r + O(r^2)$ and $Y_\tau = O(r^2)$. We finally obtain that

$$\begin{aligned} \int_{\partial D_2'} \sum_{i,j} Y_j \nu_i T_{ij} = \frac{1}{2} \int_{\partial\Omega \cap B(x_0, r)} & (x - x_0) \cdot \nu \left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right) \\ & - \frac{r}{2} \int_{\partial D_2' \cap \Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \\ & \qquad + O\left(r^2 \int_{\partial D_2' \cap \Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \right). \end{aligned}$$

Combining this with (5.25) and inserting into (5.24), we find

$$\int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2} + O\left(rE_\varepsilon(u, B(x_0,r))\right)$$
$$= -\int_{\partial\Omega\cap B(x_0,r)} (x-x_0)\cdot\nu\left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)$$
$$+ r\int_{\partial D_2'\cap\Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$
$$+ O\left(r^2 \int_{\partial D_2'\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)$$

Adding up this relation to (5.23), and observing that the contributions on $\partial\Omega$ cancel out, we are led to

$$\int_{B(x_0,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} + \int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2} + O\left(rE_\varepsilon(u, B(x_0,r))\right)$$
$$= r\int_{\partial B(x_0,r)\cap\Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$
$$+ r\int_{\partial D_2'\cap\Omega} \left(\left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right)$$
$$+ O\left(r^2 \int_{\partial D_2'\cap\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right).$$

Then, we need to do a change of variables, writing $x = \varphi(x')$. We claim that

$$\int_{D_2'} \frac{(1-|u|^2)^2}{\varepsilon^2} = \int_{D_2} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2}(1+O(r)),$$

$$\int_{\partial D_2'\cap\Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 =$$
$$\int_{\partial B(x_0,r)\cap(\mathbb{R}^2\setminus\Omega)} \left|\frac{\partial \overline{u}}{\partial \tau}\right|^2 - \left|\frac{\partial \overline{u}}{\partial \nu}\right|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}$$
$$+ O\left(r \int_{\partial D_2'\cap\Omega} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}\right).$$

Indeed, the change of variables involves the Jacobian $|\det D\varphi| = 1 + O(r)$ and modifications of the terms in ∇u. The result follows since $D\varphi$ approaches the reflection with respect to the tangent to $\partial\Omega$ at the rate r. We finally are left with

$$\int_{D_1} \frac{(1-|u|^2)^2}{\varepsilon^2} + \int_{D_2} \frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} + O\left(rE_\varepsilon(u, B(x_0,r))\right)$$
$$= r \int_{\partial B(x_0,r)\cap\Omega} \left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$
$$+ r \int_{\partial B(x_0,r)\cap(\mathbb{R}^2\setminus\Omega)} \left|\frac{\partial \overline{u}}{\partial \tau}\right|^2 - \left|\frac{\partial \overline{u}}{\partial \nu}\right|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}$$
$$+ O\left(r^2 \int_{\partial B(x_0,r)} |\nabla \overline{u}|^2 + \frac{(1-|\overline{u}|^2)^2}{2\varepsilon^2}\right).$$

Since $D_1 \cup D_2 = B(x_0, r)$, we have established (5.19). □

5.2 The Case of Ginzburg–Landau with Magnetic Field

We now consider (u, A) to be a solution to the Ginzburg–Landau equations with magnetic field (GL). The Pohozaev identity is again a direct consequence of Proposition 3.7.

Lemma 5.4. *Let (u, A) be a solution of (GL) and U be an open subset of Ω, then, for every vector field X, T_{ij} denoting the stress-energy tensor* *with magnetic field (see Definition 3.4), we have*

$$\int_{\partial U} \sum_{i,j} X_j \nu_i T_{ij} = \int_U \sum_{i,j} (\partial_i X_j) T_{ij},$$

where the indices i, j run over $1, 2$ and ν denotes the outer unit normal to ∂U.

The proof is exactly the same as for Lemma 5.1. Choosing $X = x - x_0$ and replacing the T_{ij}'s by their expressions as we did in the proof of (5.15), we find the Pohozaev identity

$$\begin{aligned} &\int_{B(x_0,r)\cap\Omega} \frac{(1-|u|^2)^2}{\varepsilon^2} - 2h^2 \\ &= \int_{\partial(B(x_0,r)\cap\Omega)} (x - x_0)\cdot\nu \left(|\nabla_A u \cdot \tau|^2 - |\nabla_A u \cdot \nu|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} - h^2 \right) \\ &\qquad - 2(x - x_0)\cdot\tau\,(\nabla_A u\cdot\tau, \nabla_A u \cdot \nu)\,. \end{aligned} \tag{5.26}$$

Using the same growing and merging of balls method, we deduce the analogue of Theorem 5.1, where we recall F_ε is defined in (4.1).

Theorem 5.2. Pohozaev ball construction — case with magnetic field. *Let (u, A) be a solution of (GL). Let $\mathcal{B}_0$ be a finite collection of disjoint closed balls and let $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$ satisfy the results of Theorem 4.2. Then, letting $r_0 = r(\mathcal{B}_0)$ and $r_1 = r(\mathcal{B}(s))$, there exists a constant $C(\Omega)$ depending only on Ω such that for any $s > 0$ such that $r_1 < C(\Omega)$, we have*

1. *For any $B \in \mathcal{B}(s)$ such that $B \subset \Omega$,*

$$\begin{aligned} &\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \\ &\qquad \geq \left(\frac{1}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} - Cr(B)F_\varepsilon(u, A, B) \right) \\ &\qquad\qquad \cdot \log\frac{r_1}{r_0} - Cr(B)F_\varepsilon(u, A, B). \end{aligned} \tag{5.27}$$

2. *There exists a finite collection of disjoint closed balls $\mathcal{B}'$ covering $\cup_{B\in\mathcal{B}(s)}B$ such that $r(\mathcal{B}') \le Cr_1$, and for every $B \in \mathcal{B}'$ such that $B \subset \Omega$, (5.27) holds, while, for every $B \in \mathcal{B}'$ intersecting $\partial\Omega$,*

$$C \int_{B\setminus\mathcal{B}_0} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \ge \left(\frac{1}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} - Cr(B)F_\varepsilon(u,A,B)\right)\log\frac{r_1}{r_0} - Cr(B)F_\varepsilon(u,A,B), \quad (5.28)$$

where C is some constant depending only on ω.

Proof. The proof follows the same lines as that of the Neumann case of Theorem 5.1.

— *Step 1: extension of* (u, A). Rather than extending u and A, we extend $|u|$ and $\nabla_A u$. We use the same mapping φ as before, which maps $\tilde{\Omega}\setminus\Omega$ to Ω. We define $\overline{|u|}(x) = |u|(\varphi(x))$ if $x \in \tilde{\Omega}\setminus\Omega$ (and $= |u|(x)$ if $x \in \Omega$), and

$$\overline{\nabla_A u}(x) = (D\varphi)^{-1}(\varphi(x))\nabla_A u(\varphi(x)) \quad \text{if } x \in \tilde{\Omega}\setminus\Omega.$$

Since $\nu \cdot \nabla_A u = 0$ on $\partial\Omega$, and $D\varphi$ is the orthogonal reflection with respect to the tangent to $\partial\Omega$ there, we find that $\nabla_A u$ extends continuously to $\tilde{\Omega}$. We also extend h by $\overline{h} = h(\varphi(x))$ in $\tilde{\Omega}\setminus\Omega$.

— *Step 2: Ball-growth.* We denote by

$$\mathcal{F}(x,r) = \frac{1}{2}\int_{B(x,r)} |\overline{\nabla_A u}|^2 + \frac{(1-\overline{|u|}^2)^2}{2\varepsilon^2}.$$

It is monotonic, and

$$r\frac{\partial\mathcal{F}}{\partial r}(x,r) = \frac{r}{2}\int_{\partial B(x,r)} |\overline{\nabla_A u}|^2 + \frac{(1-\overline{|u|}^2)^2}{2\varepsilon^2}.$$

Given the family $\{\mathcal{B}(t)\}$, $s > 0$, and $r_1 = r(\mathcal{B}(s))$, let $B \in \mathcal{B}(s)$. If $B \subset \Omega$ then, we can work in Ω without the extension, and (5.26) yields for any

$B(x_0, r) \subset B$,

$$\int_{B(x_0,r)} \frac{(1-|u|^2)^2}{\varepsilon^2} - 2h^2$$
$$= r \int_{\partial B(x_0,r)} |\nabla_A u \cdot \tau|^2 - |\nabla_A u \cdot \nu|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} - h^2.$$

We deduce

$$r \frac{\partial \mathcal{F}}{\partial r}(x,r) \geq \frac{1}{2} \int_{B(x,r)} \left(\frac{(1-|u|^2)^2}{\varepsilon^2} - 2h^2 \right) + \frac{r}{2} \int_{\partial B(x,r)} h^2.$$

Applying Proposition 4.1 as before, we find

$$\frac{1}{2} \int_{B \setminus \mathcal{B}_0} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq \int_0^s \sum_{B(x,r) \in \mathcal{B}(t) \cap B} r \frac{\partial \mathcal{F}}{\partial r}(x,r)\, dt$$
$$\geq \frac{1}{2} \int_{\mathcal{B}_0 \cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} \log \frac{r_1}{r_0}$$
$$- \int_0^s \sum_{B(x,r) \in \mathcal{B}(t) \cap B} \left(\int_{B(x,r)} h^2 - \frac{r}{2} \int_{\partial B(x,r)} h^2 \right) dt.$$

We claim that

$$\int_0^s \sum_{B(x,r) \in \mathcal{B}(t) \cap B} \left(\int_{B(x,r)} h^2 - \frac{r}{2} \int_{\partial B(x,r)} h^2 \right) dt$$
$$\leq C \left(s + \frac{1}{2} \right) r(B) \|h\|^2_{H^1(B)}$$
$$\leq C \left(\log \frac{r_1}{r_0} + 1 \right) r(B) F_\varepsilon(u, A, B). \quad (5.29)$$

This concludes the proof in the case $B \subset \Omega$. Let us now prove (5.29). Observe that if h^2 is constant over B, then the left-hand side of (5.29) is identically 0. We may thus prove the inequality with $h^2 - \tilde{h^2}$ in the left-hand side instead of h^2, where $\tilde{h^2}$ denotes the average of h^2 over B. Let us now observe that

$$\left|\int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} \int_{B(x,r)} (h^2 - \tilde{h^2})\, dt\right| \le s \int_B |h^2 - \tilde{h^2}|$$

while using Proposition 4.1 applied to $\mathcal{F}(x,r) = \int_{B(x,r)} |h^2 - \tilde{h^2}|$ we get

$$\left|\int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} \frac{r}{2} \int_{\partial B(x,r)} (h^2 - \tilde{h^2}) dt\right| \le \frac{1}{2}\int_B |h^2 - \tilde{h^2}|.$$

We deduce

$$\int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} \left(\int_{B(x,r)} h^2 - \frac{r}{2}\int_{\partial B(x,r)} h^2\right) dt \le \left(s + \frac{1}{2}\right)\int_B |h^2 - \tilde{h^2}|. \tag{5.30}$$

But, by Poincaré's inequality, we have

$$\int_B |h^2 - \tilde{h^2}| \le Cr(B)\int_B |\nabla h^2| \le Cr(B)\int_B |h||\nabla h| \le Cr(B)\int_B |\nabla h|^2 + h^2.$$

On the other hand, since (u, A) is a solution of (2.4) we have $|\nabla h| \le |\nabla_A u|$ from Lemma 3.3 and thus $\frac{1}{2}\int_B |\nabla h|^2 + h^2 \le F_\varepsilon(u, A, B)$. Inserting this into (5.30), we find (5.29).

— *Step 3: case of B intersecting $\partial\Omega$.* We claim that for any $B(x,r) \subset \tilde{\Omega}$, we have as $r \to 0$

$$\begin{aligned}
&\int_{B(x,r)} \frac{(1 - \overline{|u|}^2)^2}{\varepsilon^2} - 2\overline{h}^2 + O\left(r \int_{B(x,r)\cap\Omega} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + h^2\right) \\
&\qquad = r\int_{\partial B(x,r)} |\tau \cdot \overline{\nabla_A u}|^2 - |\nu \cdot \overline{\nabla_A u}|^2 + \frac{(1 - \overline{|u|}^2)^2}{2\varepsilon^2} - \overline{h}^2 \\
&\qquad\qquad + O\left(r^2 \int_{\partial B(x,r)\cap\Omega} |\nabla_A u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + rh^2\right).
\end{aligned}$$

The proof is similar to that of Lemma 5.3. We deduce

$$r\frac{\partial \mathcal{F}}{\partial r}(x,r) \geq \frac{1}{2}\int_{B(x,r)} \left(\frac{(1-|\overline{u}|^2)^2}{\varepsilon^2} - 2\overline{h}^2\right) + \frac{r}{2}\int_{\partial B(x,r)} \overline{h}^2$$

$$+ O\left(r^2\frac{\partial \mathcal{F}}{\partial r}(x,r) + r^2\int_{\partial B(x,r)\cap\Omega} h^2\right) + O\left(r\mathcal{F}(x,r) + r\int_{B(x,r)\cap\Omega} h^2\right).$$

Integrating between 0 and s as before, we are led to the fact that for any $B \in \mathcal{B}(s)$,

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\overline{\nabla_A u}|^2 + \frac{(1-\overline{|u|}^2)^2}{2\varepsilon^2}$$

$$\geq \frac{s}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2} - C\left(s+\frac{1}{2}\right) r(B)\int_B |\nabla\overline{h}|^2 + \overline{h}^2$$

$$-C\int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} \left(r^2\frac{\partial\mathcal{F}}{\partial r} + r^2\int_{\partial B(x,r)\cap\Omega} h^2 + r\mathcal{F}(x,r) + r\int_{B(x,r)\cap\Omega} h^2\right) dt.$$

The last error term on the right-hand side can be controlled as follows:

$$\int_0^s \sum_{B(x,r)\in\mathcal{B}(t)\cap B} \left(r^2\frac{\partial\mathcal{F}}{\partial r} + r^2\int_{\partial B(x,r)\cap\Omega} h^2 + r\mathcal{F} + r\int_{B(x,r)\cap\Omega} h^2\right) dt$$

$$\leq r(B)\int_0^s \left(\sum_{B(x,r)\in\mathcal{B}(t)\cap B} r\frac{\partial\mathcal{F}}{\partial r} + r\int_{\partial B(x,r)\cap\Omega} h^2\right) dt$$

$$+ \left(\mathcal{F}(B) + \int_{B\cap\Omega} h^2\right)\int_0^s r(\mathcal{B}(t)\cap B)\, dt$$

$$\leq 2r(B)\left(\mathcal{F}(B) + \int_{B\cap\Omega} h^2\right)$$

where the last inequality follows by applying Proposition 4.1 to $\mathcal{F}(x,r)+\int_{B(x,r)} h^2$, and from the fact that $r(\mathcal{B}(t)\cap B) = e^t r(\mathcal{B}(0)\cap B)$. On the other hand, $\int_B |\nabla \overline{h}|^2 + \overline{h}^2 \le C\int_{B\cap\Omega} |\nabla h|^2 + h^2$ and we conclude that

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\overline{\nabla_A u}|^2 + \frac{(1-\overline{|u|}^2)^2}{2\varepsilon^2}$$
$$\ge \frac{1}{2}\int_{\mathcal{B}_0\cap B} \frac{(1-|u|^2)^2}{\varepsilon^2}\log\frac{r_1}{r_0} - C\left(s+\frac{1}{2}\right) r(B)F_\varepsilon(u,A,B).$$

Modifying the collection of balls and finishing as in the proof of Theorem 5.1, we deduce that (5.28) holds. □

5.3 Applications

Once these results are known, we apply them to a collection $\mathcal{B}_0$ covering $\omega = \{x\in\Omega, |u(x)|\le 1-\delta\}$ where $\delta < 1$ may depend on ε, and we deduce the following result.

Theorem 5.3 (An upper bound for the potential term). *Let u be a solution of (5.1) or respectively (u,A) a solution of (GL), and write*

$$F_\varepsilon(|u|,\Omega) = \frac{1}{2}\int_\Omega |\nabla|u||^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} = M. \tag{5.31}$$

Then

1. *If u solves (5.1), for every r and δ such that $\frac{M}{r\delta^3}\le\frac{1}{\varepsilon^\beta}$, with $\beta<1$, there exists a finite collection of disjoint closed balls $\mathcal{B}$ with $r(\mathcal{B})\le Cr$ such that*

$$\int_{\{x\in\Omega_r, |u|\le 1-\delta\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le \frac{C}{1-\beta}\frac{E_\varepsilon(u,\mathcal{B})}{|\log\varepsilon|}$$

   where Ω_r denotes $\{x\in\Omega, \mathrm{dist}(x,\partial\Omega)\ge r\}$ and C is a universal constant.

2. *If u solves (5.1) with fixed Dirichlet boundary condition and Ω is strictly starshaped, or with Neumann boundary condition, for*

every r and δ such that $\frac{M}{r\delta^3} \le \frac{1}{\varepsilon^\beta}$, with $\beta < 1$, there exists a finite collection of disjoint closed balls $\mathcal{B}$ with $r(\mathcal{B}) \le Cr$ such that

$$\int_{\{x\in\Omega, |u|\le 1-\delta\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le \frac{C}{1-\beta} \frac{E_\varepsilon(u,\mathcal{B})}{|\log \varepsilon|}.$$

3. *If (u, A) solves (2.4), for every $r \le \frac{C}{|\log\varepsilon|}$ and δ such that $\frac{M}{r\delta^3} \le \frac{1}{\varepsilon^\beta}$, with $\beta < 1$, there exists a finite collection of disjoint closed balls $\mathcal{B}$ with $r(\mathcal{B}) \le Cr$ such that*

$$\int_{\{x\in\Omega, |u|\le 1-\delta\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \le \frac{C}{1-\beta} \frac{F_\varepsilon(u,A,\mathcal{B})}{|\log \varepsilon|}. \tag{5.32}$$

Proof. Let us first prove that

$$r\left(\{x\in\Omega, |u(x)|\le 1-\delta\}\right) \le C\frac{\varepsilon M}{\delta^3}. \tag{5.33}$$

We have the estimate $|\nabla|u|| \le \frac{C}{\varepsilon}$, which follows from Corollary 3.1 (or the analogue for (5.1)). Therefore, arguing as in [43], if $|u(x_0)| \le 1-\delta$, we have $|u(x)| \le 1-\frac{\delta}{2}$ in $B(x_0, \lambda\delta\varepsilon)$ for some well-chosen $\lambda > 0$ independent of ε and δ. We deduce that

$$\int_{B(x_0,\lambda\delta\varepsilon)} \frac{(1-|u|^2)^2}{\varepsilon^2} \ge \mu_0\delta^4 \tag{5.34}$$

for some constant $\mu_0 > 0$ independent of ε and δ. Let us consider the union of all such balls $B(x_0, \lambda\delta\varepsilon)$ over all $x_0 \in \omega = \{x \in \Omega, |u(x)| \le 1-\delta\}$, which cover ω. Extracting a Besicovitch covering, we may assume that each point is in at most 3 such balls, and we deduce

$$n\mu_0\delta^4 \le C\int_\Omega \frac{(1-|u|^2)^2}{\varepsilon^2} \le CM$$

where n is the number of the balls in the covering. We deduce that $n \le \frac{CM}{\delta^4}$ and thus $r(\omega)$ is bounded by the total perimeter of the balls hence by $Cn\lambda\delta\varepsilon \le \frac{CM\varepsilon}{\delta^3}$.

For the case of (5.1), we directly apply Theorem 5.1 to $\mathcal{B}_0$, a finite collection of disjoint balls covering $\omega = \{x \in \Omega, |u(x)| \le 1-\delta\}$, and s

such that $r(\mathcal{B}(s)) = r$. From (5.33), we have $r(\omega) < C\frac{\varepsilon M}{\delta^3}$, hence we can have $r(\mathcal{B}_0) < C\frac{\varepsilon M}{\delta^3}$. Theorem 4.2 yields a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}_+}$, and given r small enough, Theorem 5.1 allows us to construct from $\mathcal{B}(s)$ (such that $r(\mathcal{B}(s)) = r$) a finite family of disjoint closed balls $\mathcal{B}$ covering $\mathcal{B}_0$ such that $r(\mathcal{B}) \le Cr$. Restricting to $\mathcal{B}_0 \cap \Omega_r$ instead of $\mathcal{B}_0$, we are sure that all the balls in the collection are in a tubular neighborhood of size r of $\cup_{B\in\mathcal{B}_0} B$ and hence are included in Ω. We may thus apply the estimates (5.3) and add them up over all the balls in $\mathcal{B}$, getting

$$E_\varepsilon(u,\mathcal{B}) \ge \left(\frac{1}{2}\int_{\omega\cap\Omega_r} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)\log\frac{r\delta^3}{C\varepsilon M}$$

hence in view of the assumptions on r and δ,

$$\begin{aligned} E_\varepsilon(u,\mathcal{B}) &\ge \left(\frac{1}{2}\int_{\omega\cap\Omega_r} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)\left(\log\frac{1}{\varepsilon} - \log\frac{CM}{r\delta^3}\right) \\ &\ge \left(\frac{1}{2}\int_{\omega\cap\Omega_r} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)((1-\beta)|\log\varepsilon| - C) \end{aligned}$$

and the result easily follows. The Dirichlet case works exactly the same way, using (5.4) instead of (5.3).

For the case of (GL), apply Theorem 5.2 with $\mathcal{B}_0$ covering the same ω. This yields a finite collection of disjoint closed balls $\mathcal{B}$ with $r(\mathcal{B}) = r$ such that, adding up over all the balls the estimates found in (5.27) or (5.28), we have

$$\begin{aligned} CF_\varepsilon(u,A,\mathcal{B}) \ge \left(\frac{1}{2}\int_{\omega} \frac{(1-|u|^2)^2}{\varepsilon^2} - Cr(\mathcal{B})F_\varepsilon(u,A,\mathcal{B})\right)\log\frac{r}{r(\omega)} \\ - Cr(\mathcal{B})F_\varepsilon(u,A,\mathcal{B}). \end{aligned}$$

Arguing as above, we deduce

$$CF_\varepsilon(u,A,\mathcal{B}) \ge \left(\frac{1}{2}\int_{\omega} \frac{(1-|u|^2)^2}{\varepsilon^2} - Cr(\mathcal{B})F_\varepsilon(u,A,\mathcal{B})\right)\big((1-\beta)|\log\varepsilon| - C\big)$$

and if $r(\mathcal{B}) \le \frac{C}{|\log\varepsilon|}$ we conclude that (5.32) holds. $\square$

As a main application, we obtain that if u is a solution of (5.1) with Dirichlet boundary condition and $E_\varepsilon(u) \leq C|\log\varepsilon|$, then $\int_{|u|\leq\frac{1}{2}}(1-|u|^2)^2 \leq C$. This in turn suffices to bound by a uniform constant the number of vortices of u. Then, below we use Theorem 5.1 again to get improved lower bounds in terms of the degrees of the vortices. In order to obtain an analogous result for the situation with magnetic field, we need to anticipate a bit on the forthcoming chapters, and introduce h_0 the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 \text{ in } \Omega \\ \qquad\quad h_0 = 1 \text{ on } \partial\Omega. \end{cases}$$

Once h_0 is defined, for any A, we define $A' = A - h_{\mathrm{ex}}\nabla^\perp h_0$.

Theorem 5.4 (Microscopic lower bound). *Let $\{u_\varepsilon\}_\varepsilon$ be solutions of (5.1), such that $E_\varepsilon(u_\varepsilon) \leq C|\log\varepsilon|$, with Dirichlet boundary condition (and Ω strictly starshaped) or Neumann boundary condition, or let respectively $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be solutions of (GL) such that $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \leq C|\log\varepsilon|$ and $h_{ex} \leq \varepsilon^{-\beta}$ with $\beta < 1$. Then the following holds as $\varepsilon \to 0$.*

For every $\eta > 0$, there exists $R > 0$ and for any ε small enough a finite collection of disjoint balls $B(a_1, R\varepsilon), \dots, B(a_k, R\varepsilon)$ (a_i depending on ε) with k bounded independently of ε such that

1. *$\{|u_\varepsilon| \leq \frac{1}{2}\} \subset \cup_{i=1}^k B(a_i, R\varepsilon)$.*
2. *$|a_i - a_j| \gg \varepsilon$ for $i \neq j$, and $dist(a_i, \partial\Omega) \gg \varepsilon$ for every i.*
3. *The $d_i = \deg(u_\varepsilon, \partial B(a_i, R\varepsilon))$ are all nonzero.*
4. *For every $1 > r \gg \varepsilon$,*

$$E_\varepsilon(u_\varepsilon) \geq \sum_{\substack{i\in[1,k] \\ \mathrm{dist}(a_i,\partial\Omega)\geq r}} (\pi d_i^2 - \eta)\log\frac{r}{C\varepsilon} \tag{5.35}$$

respectively for any $r \ll \min\left(|\log\varepsilon|^{-1}, (\sqrt{|\log\varepsilon|}h_{ex})^{-1}\right)$,

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \geq \sum_{\substack{i\in[1,k] \\ \mathrm{dist}(a_i,\partial\Omega)\geq r}} (\pi d_i^2 - \eta)\log\frac{r}{C\varepsilon} - o(1). \tag{5.36}$$

Moreover, if u_ε (resp. $(u_\varepsilon, A_\varepsilon)$) is a very local minimizer (as in Definition 3.8) of E_ε, resp. G_ε, around any point, then $\forall i B(a_i, R\varepsilon)$ contains a unique zero of u_ε of degree $d_i = \pm 1$ in 3).

Before we give the proof let us state a simple lemma, a consequence of Theorem 3.4.

Lemma 5.5. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions of (5.1) with $E_\varepsilon(u_\varepsilon) \ll \frac{1}{\varepsilon^2}$, respectively $(u_\varepsilon, A_\varepsilon)$ solutions of (GL) with $F_\varepsilon(u_\varepsilon, A_\varepsilon) \ll \frac{1}{\varepsilon^2}$, in a domain Ω. For every $c > 0$ and every $\eta > 0$ there exists $R > c$ such that if $|u_\varepsilon| \geq \frac{1}{2}$ in $B(x_\varepsilon, R\varepsilon)\backslash B(x_\varepsilon, c\varepsilon)$, letting $d_\varepsilon = \deg(u_\varepsilon, \partial B(x_0, c\varepsilon))$, we have*

$$\int_{B(x_\varepsilon,R\varepsilon)} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \geq 2\pi d_\varepsilon^2 - \eta.$$

Proof. We start by observing that the degree d_ε is bounded independently of ε. Indeed, recall the definition

$$d = \frac{1}{2\pi} \int_{\partial B(x_\varepsilon,c\varepsilon)} \frac{1}{|u|^2} \left(iu, \frac{\partial u}{\partial \tau} \right).$$

With the a priori bounds for solutions $|u_\varepsilon| \leq 1$ and $|\nabla u_\varepsilon| \leq \frac{C}{\varepsilon}$ resp. $|\nabla_{A_\varepsilon} u_\varepsilon| \leq \frac{C}{\varepsilon}$ (see Corollary 3.1), we easily deduce that d_ε is bounded independently of ε.

If the desired property were not true, this would mean that we can find $\eta > 0$, $c > 0$, and a sequence u_ε of such solutions and of points x_ε, such that for every R, $|u_\varepsilon| \geq \frac{1}{2}$ in $B(x_\varepsilon, R\varepsilon)\backslash B(x_0, c\varepsilon)$, and

$$\int_{B(x_\varepsilon,R\varepsilon)} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leq 2\pi d_\varepsilon^2 - \eta \tag{5.37}$$

where $d_\varepsilon = \deg(u_\varepsilon, \partial B(x_\varepsilon, c\varepsilon))$. Since we saw that d_ε remains bounded, we deduce that $\int_{B(x_\varepsilon,R\varepsilon)} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leq C$ where C is independent of ε and R. Rescaling and considering $U_\varepsilon(x) = u_\varepsilon(x_\varepsilon + \varepsilon x)$, from Proposition 3.12 we find that, after extraction of a subsequence, U_ε converges in $C^1_{loc}(\mathbb{R}^2)$ to U, the solution of (3.12), with $|U| \geq \frac{1}{2}$ in $\mathbb{R}^2\backslash B(0, c)$ and $\int_{\mathbb{R}^2}(1-|U|^2)^2 \leq C$. Indeed, $\int_{B(x_\varepsilon,R\varepsilon)} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} = \int_{B(0,R)}(1-|U_\varepsilon|^2)^2 \leq C$, so by strong C^1_{loc} convergence $\int_{B(0,R)}(1-|U|^2)^2 \leq C$ and since this is true for every R and C is independent of R, we deduce that $\int_{\mathbb{R}^2}(1-|U|^2)^2 \leq C$.

Moreover, we may assume $d_\varepsilon \to d = deg(U, \partial B(0, c))$ (because the d_ε form a bounded sequence of integers). From Theorem 3.4, we find that

$\int_{\mathbb{R}^2}(1-|U|^2)^2 = 2\pi d^2$. By the strong convergence of U_ε this implies that

$$\int_{B(x_\varepsilon,R\varepsilon)} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} - 2\pi d_\varepsilon^2 \geq -\eta/2,$$

for ε small enough and R large enough, a contradiction with (5.37). $\square$

Proof of Theorem 5.4.
— *Step 1: Boundedness of the potential.* For the case without magnetic field, we deduce from Theorem 5.3, combined with the bound $E_\varepsilon(u) \leq C|\log\varepsilon|$, that

$$\int_{\{x\in\Omega,|u|\leq 1-\frac{1}{|\log\varepsilon|}\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C. \tag{5.38}$$

In the case with magnetic field, we observe that since h_0 is a smooth function, for any collection of disjoint closed balls, we have

$$\begin{aligned} F_\varepsilon(u,A,\mathcal{B}) &\leq F_\varepsilon(u,A') + C{h_{\text{ex}}}^2\sum_{B\in\mathcal{B}} r(B)^2 + Ch_{\text{ex}}\sum_{B\in\mathcal{B}} r(B)\sqrt{F_\varepsilon(u,A')} \\ &\leq F_\varepsilon(u,A') + C{h_{\text{ex}}}^2 r(\mathcal{B})^2 + Ch_{\text{ex}} r(\mathcal{B})\sqrt{F_\varepsilon(u,A')}. \end{aligned} \tag{5.39}$$

Also observe that $M \leq F_\varepsilon(u,A') \leq C|\log\varepsilon|$ in (5.31). So choosing in Theorem 5.3 $r = \min(\frac{|\log\varepsilon|^{\frac{1}{2}}}{h_{\text{ex}}}, \frac{1}{|\log\varepsilon|})$, we also find that

$$\int_{\{|u|\leq 1-\frac{1}{|\log\varepsilon|}\}} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C.$$

— *Step 2: Boundedness of the number of balls and properties of the balls.* This step is as in [43] and [77]. From (5.34) applied to $\delta = \frac{1}{2}$, if $|u(x_0)| \leq \frac{1}{2}$, then

$$\int_{B(x_0,\lambda\varepsilon)} \frac{(1-|u|^2)^2}{\varepsilon^2} \geq \mu_0 \tag{5.40}$$

for some $\mu > 0$ and $\lambda > 0$ independent of ε. Thus, combining this with (5.38), we see that there can only be a uniformly bounded number of disjoint balls of radius $\lambda\varepsilon$ which intersect $\{|u| \le \frac{1}{2}\}$. Using a covering argument as in [43], we deduce that the set $\{|u| \le \frac{1}{2}\}$ can be covered by a finite number of disjoint balls of radius $\lambda\varepsilon$ centered at a_i, the number of balls remaining bounded independently of ε. Merging the balls into larger balls if necessary we can always assume that $|a_i - a_j| \gg \varepsilon$ for $i \neq j$. Moreover, we have $\operatorname{dist}(a_i, \partial\Omega) \gg \varepsilon$, because otherwise (5.40) would be in contradiction with the last part of Proposition 3.12 (valid, as we mentioned, without magnetic field).

Finally, let us assume by contradiction that $d_i = \deg(u, \partial B(a_i, \lambda\varepsilon)) = 0$. We may assume $|u(a_i)| \le \frac{1}{2}$ (otherwise the ball could be removed from the collection) and, considering $U_\varepsilon(x) = u(a_i + \varepsilon x)$, we may assume that U_ε converges in $C^1_{loc}(\mathbb{R}^2)$ to a solution U of (3.12), of total degree 0 on large circles. Passing to the limit in (5.38), we find that $\int_{\mathbb{R}^2}(1-|U|^2)^2 < \infty$. It is known (see [61]) that such solutions of degree 0 are constants of modulus 1. This is in contradiction with $|u(a_i)| \le \frac{1}{2}$ and the uniform convergence of U_ε. Thus, the degrees d_i are all nonzero.

— *Step 3: Lower bound.* We may now apply Theorems 5.1 or 5.2 to $\mathcal{B}_0 = \{B(a_i, R\varepsilon), i \in [1,k], \operatorname{dist}(a_i, \partial\Omega) \ge 4r\}$ and s such that $r(\mathcal{B}(s)) = r$. This ensures that the balls we obtain, being of radius less than r, do not intersect $\{x \in \Omega, \operatorname{dist}(x, \partial\Omega) \le r\}$. Combining the result of Theorems 5.1 to the result of Lemma 5.5, we find that (5.35) hold.

For (5.36), combining the result of Theorem 5.2 to Lemma 5.5, we find

$$F_\varepsilon(u, A, \mathcal{B}) \ge \left(\pi \sum_i d_i^2 - \eta - CrF_\varepsilon(u, A, \mathcal{B})\right) \log \frac{r}{C\varepsilon} - CrF_\varepsilon(u, A, \mathcal{B}).$$

Choosing $r \ll \frac{1}{|\log\varepsilon|}$ and $r \ll \frac{1}{h_{\rm ex}\sqrt{|\log\varepsilon|}}$ and using (5.39), we have $F_\varepsilon(u, A, \mathcal{B}) \le F_\varepsilon(u, A') + o(1)$ and $rF_\varepsilon(u, A, \mathcal{B}) \le o(1)$ hence

$$F_\varepsilon(u, A') \ge \left(\pi \sum_i d_i^2 - \eta\right) \log \frac{r}{C\varepsilon} - o(1)$$

and (5.36) is proved.

The last assertion follows from Proposition 3.12. □

Remark 5.2. 1. Our results (5.35)–(5.36) are easy consequences of Theorems 5.1 and 5.2. A stronger result is proved by Comte and Mironescu in [77, 79], using more specific arguments: for solutions of (5.1) with Dirichlet boundary conditions, the equality $E_\varepsilon(u) = \pi \sum_i d_i^2 |\log \varepsilon| + O(1)$ holds.

2. By a diagonal argument, one can obtain $o(1)$ instead of η in the lower bounds above.

3. In the case of (5.1) with Dirichlet boundary condition, it was proved in [43] that
$$\frac{(1-|u|^2)^2}{\varepsilon^2} \rightharpoonup 2\pi \sum_{i=1}^k d_i^2 \delta_{a_i^0}$$
in the sense of measures, where the a_i^0's are the limits of the vortex-points a_i^ε as $\varepsilon \to 0$, and belong to Ω. Therefore, it means that $\frac{(1-|u|^2)^2}{\varepsilon^2} \rightharpoonup 0$ in the sense of measures in a neighborhood of $\partial\Omega$, hence there can be no a_i^ε above converging to $\partial\Omega$ because it would contradict (5.40). Thus the condition $\mathrm{dist}(a_i, \partial\Omega) \geq C > 0$ for small ε is always satisfied in that case, and one may take $r = \min(C, \frac{1}{2})$ in the theorem above.

The previous theorem does not apply to unbounded numbers of vortices. However, we may return to the setting of Chapter 4 and link the regular ball-construction with this Pohozaev ball-construction. In the same spirit as Theorem 5.4, this yields details on the microscopic behavior of local minimizers in the case of an unbounded number of vortices.

In the next propositions, we take advantage of the fact that the lower bounds of Propositions 4.2 and 4.3 really include the squares of the degrees, to say that if the energy grows like the total degree times log during the ball growth, then the degrees at appropriate small scales should be ± 1.

In what follows, as in Chapter 4, if B is a ball, d_B denotes the degree of the map on the boundary of the ball if $B \subset \Omega$, and 0 otherwise.

Proposition 5.1. (Microscopic analysis of very local minimizers — case without magnetic field). *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be very local minimizers, in the sense of Definition 3.8, of E_ε (around every point), and such that $E_\varepsilon(u_\varepsilon) \ll \frac{1}{\varepsilon^2}$. Let $\mathcal{B}(s)$ be a collection of disjoint closed balls obtained by ball growth from an initial collection $\mathcal{B}_0$, as in Theorem 4.2; such that, as $\varepsilon \to 0$,*

1. $s \geq \beta |\log \varepsilon|$ for some $1 \geq \beta > 0$.

2. *There exists* $\delta = o(1)$ *such that*

$$|u_\varepsilon| \geq 1 - \delta \quad \textit{in } \Omega \setminus \cup_{B \in \mathcal{B}_0} B, \tag{5.41}$$

and $\delta E_\varepsilon(u_\varepsilon) \leq o(|\log \varepsilon|)$.

3. *Where* $D = \sum_{B \in \mathcal{B}(s)} |d_B|$,

$$E_\varepsilon(u_\varepsilon, \mathcal{B}(s)) - E_\varepsilon(u_\varepsilon, \mathcal{B}_0) \leq \pi D s + o(|\log \varepsilon|). \tag{5.42}$$

Then, for ε *small enough, in the union of the balls of* $\mathcal{B}(s)$ *that do not intersect* $\partial\Omega$, u_ε *has exactly* D *zeroes, more precisely each* $B \in \mathcal{B}(s)$ *such that* $B \subset \Omega$ *contains exactly* $|d_B|$ *zeroes, all of degree* $\pm 1 = sign(d_B)$.

Proof. — *Step 1: Use of the ball-construction.*
Let v denote $\frac{u_\varepsilon}{|u_\varepsilon|}$ in $\Omega \setminus \cup_{B \in \mathcal{B}_0} B$. Let $\mathcal{B}(t)$ be the collection of balls in the ball growth for $t \in [0, s]$. For every $B \in \mathcal{B}(s)$, we recall the notation

$$\|D_B\|^2(t) = \sum_{\substack{B' \in \mathcal{B}(t) \cap B \\ B' \subset \Omega}} d_{B'}^2.$$

From Proposition 4.2, we have

$$\frac{1}{2} \int_{B \setminus \mathcal{B}(0)} |\nabla v|^2 \geq \pi \int_0^s \|D_B\|^2(t)\, dt. \tag{5.43}$$

Summing over all $B \in \mathcal{B}(s)$, and comparing with (5.41) and (5.42), we deduce

$$\begin{aligned} \pi \sum_{B \in \mathcal{B}(s)} \int_0^s \|D_B\|^2(t)\, dt &\leq (1 + 2\delta) \sum_{B \in \mathcal{B}(s)} \left(E_\varepsilon(u_\varepsilon, B) - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B)\right) \\ &\leq \pi s \sum_{B \in \mathcal{B}(s)} |d_B| + o(|\log \varepsilon|). \end{aligned} \tag{5.44}$$

On the other hand, we always have (see Lemma 4.2), for every $B \in \mathcal{B}(s)$,

$$\|D_B\|^2(t) \geq |d_B| \tag{5.45}$$

with equality if and only if $d_{B'} = sign(d_B)$ (or 0) for every $B' \in \mathcal{B}(t) \cap B$. If there is not equality, since the inequality involves integers, we have $\|D_B\|^2(t) \geq |d_B| + 1$. Let us assume by contradiction that, given $0 < \alpha_1 < \alpha_2 \le \beta$, for every $t \in [\alpha_1|\log\varepsilon|, \alpha_2|\log\varepsilon|]$, there exists a ball $B \in \mathcal{B}(s)$ such that we have $\|D_B\|^2(t) \neq |d_B|$, hence $\|D_B\|^2(t) \ge |d_B|+1$. Plugging this into (5.44), we find $(\alpha_2-\alpha_1)|\log\varepsilon| \le o(|\log\varepsilon|)$, a contradiction. Hence, for every $0 < \alpha_1 < \alpha_2 \le \beta$, there exists $t \in [\alpha_1|\log\varepsilon|, \alpha_2|\log\varepsilon|]$ such that for all $B \in \mathcal{B}(s)$, $\|D_B\|^2(t) = |d_B|$ and thus $d_{B'} = \deg(v, \partial B') = sign(d_B)$ (or 0) for every $B' \in \mathcal{B}(t) \cap B$. In other words, picking such a t, there are $|d_B|$ balls of nonzero degree in the collection $\mathcal{B}(t) \cap B$, they all have degree $d_{B'} = sign(d_B)$, and

$$\sum_{B' \in \mathcal{B}(t)\cap B} d_{B'} = d_B.$$

Moreover, comparing (5.45) with (5.44), we must have for every $B \in \mathcal{B}(s)$,

$$E_\varepsilon(u_\varepsilon, B) - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B) = \pi|d_B|s + o(|\log\varepsilon|). \tag{5.46}$$

On the other hand, using the ball-construction as above, we have, for every $B' \in \mathcal{B}(t) \cap B$,

$$E_\varepsilon(u_\varepsilon, B') - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B') \ge \pi|d_{B'}|t + o(|\log\varepsilon|), \tag{5.47}$$

while

$$E_\varepsilon(u_\varepsilon, B) - E_\varepsilon(u_\varepsilon, \mathcal{B}(t) \cap B) + o(|\log\varepsilon|) \ge \int_t^s \|D_B\|^2(k)\,dk \ge \pi|d_B|(s-t) \tag{5.48}$$

where we have used (5.45). Combining (5.48) to (5.46), we find

$$E_\varepsilon(u_\varepsilon, \mathcal{B}(t) \cap B) - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B) \le \pi|d_B|t + o(|\log\varepsilon|).$$

Comparing this to (5.47) which we sum over all $B' \subset B$, we have

$$\pi \sum_{B' \in \mathcal{B}(t) \subset B} |d_{B'}|t \le E_\varepsilon(u_\varepsilon, \mathcal{B}(t) \cap B) - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B) + o(|\log\varepsilon|)$$
$$\le \pi|d_B|t = \pi \sum_{B' \in \mathcal{B}(t) \subset B} |d_{B'}|t.$$

We deduce that there must be equality for each B', that is, for every $B' \in \mathcal{B}(t)$,

$$E_\varepsilon(u_\varepsilon, B') - E_\varepsilon(u_\varepsilon, \mathcal{B}_0 \cap B') = \pi |d_{B'}| t + o(|\log \varepsilon|) = \pi t + o(|\log \varepsilon|) \tag{5.49}$$

(or $= o(|\log \varepsilon|)$ if $d_{B'} = 0$) since, as we established, $d_{B'} = sign(d_B)$.

— *Step 2: Use of the Pohozaev ball-construction.*

We still consider a $t \gg 1$ which satisfies the conclusions of Step 1. Theorem 5.1 applied with $\mathcal{B}_0$ and $\mathcal{B}(t)$ yields, for every $B' \in \mathcal{B}(t)$ such that $B' \subset \Omega$,

$$\frac{t}{2} \int_{B' \cap \mathcal{B}_0} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \le \frac{1}{2} \int_{B' \setminus \mathcal{B}_0} |\nabla u_\varepsilon|^2 + \frac{(1 - |u_\varepsilon|^2)^2}{2\varepsilon^2},$$

hence from (5.49),

$$\frac{1}{2} \int_{B' \cap \mathcal{B}_0} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \le \pi + o(1), \tag{5.50}$$

or $\le o(1)$ if $d_{B'} = 0$.

— *Step 3: Blow-up analysis.*

Now let a_i^ε be the zeroes of u_ε. According to Proposition 3.13, since $E_\varepsilon(u_\varepsilon) \ll \frac{1}{\varepsilon^2}$, for any such a_i^ε, the rescaled maps $w_\varepsilon(x) = u_\varepsilon(a_i^\varepsilon + \varepsilon x)$ converge as $\varepsilon \to 0$, up to extraction, to a radial solution of (3.12) as described in Theorem 3.2, i.e., a solution with a unique zero of degree $+1$ or -1. We deduce that any two zeroes of u_ε are at a distance $\gg \varepsilon$ from each other and from the boundary. Moreover, from Theorem 3.4, we have

$$\lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{B(a_i^\varepsilon, R\varepsilon)} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} = 2\pi,$$

and we deduce as in Lemma 5.5 that for every $\eta > 0$, there exists $R > 0$ such that the $B(a_i^\varepsilon, R\varepsilon)$'s are disjoint, and, using the notation above,

$$\int_{B' \cap \mathcal{B}_0} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \ge \int_{\cup_{i / a_i^\varepsilon \in B'} B(a_i^\varepsilon, R\varepsilon) \cap \{|u| \le 1 - \delta\}} \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \ge \sum_{i / a_i^\varepsilon \in B'} 2\pi - \eta.$$

Comparing with (5.50), we immediately deduce

$$\sum_{i/a_i^\varepsilon \in B'} (2\pi - \eta) \le 2\pi$$

hence, choosing $\eta < 1$, we find that each $B' \in \mathcal{B}(t) \cap B$ which does not intersect $\partial\Omega$ contains at most one zero of u, of degree $sign(d_B)$. Since, for every $B \in \mathcal{B}(s)$ included in Ω, $\mathcal{B}(t) \cap B$ contains exactly $|d_B|$ balls of nonzero degree, we find that B contains exactly $|d_B|$ zeroes of degree $sign(d_B)$, hence the result. □

The following version with magnetic field will be used in Chapter 11.

Proposition 5.2. (Microscopic analysis of very local minimizers — case with magnetic field). *Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be very local minimizers of G_ε (around every point) in the sense of Definition 3.8. Let $\mathcal{B}(s)$ be a collection of disjoint closed balls obtained by ball growth from an initial collection $\mathcal{B}_0$, as in Theorem 4.2; as $\varepsilon \to 0$,*

1. *$s \ge \beta|\log\varepsilon|$ for some $1 \ge \beta > 0$.*
2. *There exists $\beta' < \beta$ such that*

$$F_\varepsilon(u_\varepsilon, A_\varepsilon) \le \frac{1}{\varepsilon^{\beta'}}. \tag{5.51}$$

3. *There exists $\delta = o(1)$ such that*

$$|u_\varepsilon| \ge 1 - \delta \quad in\ \Omega \setminus \cup_{B \in \mathcal{B}_0} B, \tag{5.52}$$

   and $\delta F_\varepsilon(u_\varepsilon, A_\varepsilon) \le o(|\log\varepsilon|)$.

4. *Where $D = \sum_{B\in\mathcal{B}(s)} |d_B|$,*

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}(s)) - F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_0) \le \pi D s + o(|\log\varepsilon|). \tag{5.53}$$

Then, for ε small enough, in the union of the balls of $\mathcal{B}(s)$ that do not intersect $\partial\Omega$, u_ε has exactly D zeroes, more precisely each $B \in \mathcal{B}(s)$ such that $B \subset \Omega$ contains exactly $|d_B|$ zeroes, all of degree $\pm 1 = sign(d_B)$.

Proof. The proof is along the same lines as for the case without magnetic field. We present the main adjustments that need to be made.

In the first step, we replace (5.43) by (4.14) which, denoting $r_1 = r(\mathcal{B}(s))$ and $r_0 = r(\mathcal{B}_0)$, yields

$$\frac{1}{2}\int_{B\setminus\mathcal{B}_0} |\nabla_A v|^2 + \frac{1}{2} r(B)(r_1 - r_0)\int_B |\operatorname{curl} A|^2 \\ \geq \pi \int_0^s \|D_B\|^2(t)\left(1 - \frac{r(\mathcal{B}(t))}{r_1 - r_0}\right) dt.$$

From (5.53), we deduce by (5.52) that

$$\pi D s + o(|\log\varepsilon|) \geq \frac{1}{2}\sum_{B\in\mathcal{B}(s)}\int_{B\setminus\mathcal{B}_0} |\nabla_A v|^2 + |\operatorname{curl} A|^2 \\ \geq \pi\int_0^s \|D_B\|^2(t)\left(1 - \frac{r(\mathcal{B}(t))}{r_1 - r_0}\right) dt.$$

Assume by contradiction that there exists $0 < \alpha_1 < \alpha_2 < \beta$ such that for $t \in [\alpha_1|\log\varepsilon|, \alpha_2|\log\varepsilon|]$, we have $\|D_B(t)\|^2 \geq d_B + 1$, then, since $\alpha_2 < \beta$, we have $r(\mathcal{B}(t)) \ll r_1$ in this interval; thus we find, arguing as in the case without magnetic field,

$$o(|\log\varepsilon|) \geq \int_{\alpha_1|\log\varepsilon|}^{\alpha_2|\log\varepsilon|} \left(\|D_B\|^2(t)(1 - o(1)) - d_B\right) dt \geq \frac{1}{2}(\alpha_2 - \alpha_1)|\log\varepsilon|,$$

a contradiction. The rest of the step follows as in the case without magnetic field.

For the second step, we pick $t \leq \alpha|\log\varepsilon|$ with $\alpha < \beta - \beta'$, and use Theorem 5.2 which yields, if $B' \subset \Omega$,

$$\frac{t}{2}\int_{B'\cap\mathcal{B}_0} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \leq \frac{1}{2}\int_{B'\setminus\mathcal{B}_0} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \\ + C(t+1) r(B') F_\varepsilon(u_\varepsilon, A_\varepsilon, B') \\ \leq \pi t + o(|\log\varepsilon|)$$

Now, $r(B') \le r(\mathcal{B}(s))e^{t-s} \le e^{t-s} \le \varepsilon^{\beta-\alpha} \ll \varepsilon^{\beta'}$, thus $r(B')F_\varepsilon(u_\varepsilon, A_\varepsilon, B')$ $\ll 1$ from (5.51), hence we find

$$\frac{t}{2} \int_{B' \cap \mathcal{B}_0} \frac{(1-|u_\varepsilon|^2)^2}{\varepsilon^2} \le \pi t + o(|\log \varepsilon|)$$

and $o(|\log \varepsilon|)$ if $d_{B'} = 0$, and thus the same result (5.50) holds. The third step is the same. □

Bibliographic notes on Chapter 5: The results of this chapter and the idea of coupling the ball construction method to Pohozaev are new; however, as we mentioned, the Pohozaev identity has always been used for Ginzburg–Landau starting with Bethuel–Brezis–Hélein, Brezis–Merle–Rivière [43, 61] and Bethuel–Rivière [52], Struwe [189] on small balls, in particular for deducing bounds on the potential from bounds on the energy. In this respect, Bethuel–Orlandi–Smets [50] have some related and general result valid in any dimension.

Chapter 6

Jacobian Estimate

In this chapter we show that the vortex balls provided by Theorem 4.1, although they are constructed through a complicated process and are not completely intrinsic to (u, A) (and not unique), have in the end a simple relation to the configuration (u, A), namely that the measure $\sum_i 2\pi d_i \delta_{a_i}$ is close in a certain norm to the gauge-invariant version of the Jacobian determinant of u, an intrinsic quantity depending on (u, A). This will allow us, in the next chapters, to extract from $G_\varepsilon(u, A)$, in addition to the vortex energy $\pi \sum_i |d_i||\log \varepsilon|$ contained in the vortex balls, a term describing vortex-vortex interactions and vortex-applied field interactions in terms of the measure $\sum_i 2\pi d_i \delta_{a_i}$.

The results of this chapter are used throughout the remainder of the book, in the form of Theorems 6.1 and 6.2.

Notation: For $u : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$ we let

$$\mu(u, A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A. \tag{6.1}$$

$\mu(u, A)$ will most often be abbreviated in μ. This is a gauge-invariant quantity that will play the role of the Jacobian determinant of u ($d(u \times du)$ when $A = 0$). Again, it suffices to set $A = 0$ below to get the corresponding result relating the Jacobian to (1.2).

For any domain Ω and $\varepsilon > 0$ we let again

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Definition 6.1. For $\beta \in (0, 1]$ we let $C_0^{0,\beta}(\Omega)$ denote the space of functions in $C^{0,\beta}(\Omega)$ that vanish on the boundary and $(C_0^{0,\beta}(\Omega))^*$ its dual.

We use as a norm for $f \in C_0^{0,\beta}(\Omega)$ the quantity

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta},$$

and the standard dual norm on $(C_0^{0,\beta}(\Omega))^*$. Note that in the case $\beta = 1$, the norm on $C_0^{0,\beta}(\Omega)$ is simply the Lipschitz norm.

Theorem 6.1. *Let $u : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$ be C^1, let $\mathcal{B} = \{B_i\}_{i\in I}$ be a finite collection of disjoint closed balls and let $\varepsilon > 0$ be such that*

$$\{x \in \Omega_\varepsilon, ||u(x)| - 1| \geq 1/2\} \subset \cup_i B_i.$$

Then, letting $r = r(\mathcal{B})$ and $M = F_\varepsilon(u, A)$, and defining μ by (6.1) *we have, assuming $\varepsilon, r \leq 1$,*

$$\left\| \mu - 2\pi \sum_{\substack{i\in I\\ B_i\subset\Omega_\varepsilon}} d_i \delta_{a_i} \right\|_{(C_0^{0,1}(\Omega))^*} \leq C \max(r,\varepsilon)(1+M). \tag{6.2}$$

where a_i is the center of B_i, $d_i = \deg(u/|u|, \partial B_i)$, and C is a universal constant.

Moreover, using the same notation,

$$\|\mu\|_{(C^0)^*} \leq CM. \tag{6.3}$$

This result was proved by Jerrard–Soner in [119] under a slightly different form. The proof we present here is closer to that of a result of similar nature we obtained in [169, 168].

6.1 Preliminaries

Definition 6.2. We define $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ as follows.

If $x \in [0, 1/2]$, then $\chi(x) = 2x$. If $x \in [1/2, 3/2]$, then $\chi(x) = 1$. If $x \in [3/2, 2]$ then $\chi(x) = 1 + 2(x - 3/2)$. Finally if $x \geq 2$, then $\chi(x) = x$.

We have:

Lemma 6.1. *For any* $t \in \mathbb{R}_+$ *the function* χ *above satisfies*

$$\chi(t) \le 2t, \quad \chi'(t) \le 2, \quad |\chi(t) - t| \le |1 - t|, \quad |\chi(t) - 1| \le |1 - t| \tag{6.4}$$

and

$$\left|\chi(t)^2 - t^2\right| \le 3t|1 - t|. \tag{6.5}$$

Proof. Properties in (6.4) follow directly by inspecting the graph of the function χ while (6.5) follows by noting that

$$\left|\chi(t)^2 - t^2\right| = |\chi(t) + t|\,|\chi(t) - t| \le 3t|1 - t|. \qquad \square$$

To prove the theorem, we assume its hypotheses are satisfied for some (u, A), some collection of balls $\mathcal{B}$ and some $\varepsilon > 0$. We define

$$\rho = |u|, \quad \tilde{u} = \frac{\chi(\rho)}{\rho} u, \quad \tilde{\mu} = \operatorname{curl}(i\tilde{u}, \nabla_A \tilde{u}) + \operatorname{curl} A. \tag{6.6}$$

Observe that the main point of this construction is that $|\tilde{u}| = 1$ wherever $|u|$ is close enough to 1.

We claim that:

Lemma 6.2. *For some universal constant* $C > 0$ *we have*

$$\|\mu - \tilde{\mu}\|_{(C_0^{0,1}(\Omega))^*} \le C\varepsilon M.$$

Proof. Let $j = (iu, \nabla_A u)$ and $\tilde{j} = (i\tilde{u}, \nabla_A \tilde{u})$. Then for any function $\zeta \in C_0^{0,1}(\Omega)$ we have

$$\left|\int_\Omega \zeta(\mu - \tilde{\mu})\right| = \left|\int_\Omega \nabla^\perp \zeta \cdot (j - \tilde{j})\right| \le \|\nabla\zeta\|_\infty \|j - \tilde{j}\|_{L^1(\Omega)}. \tag{6.7}$$

But, writing $u = \rho e^{i\varphi}$ and $\tilde{u} = \tilde{\rho} e^{i\varphi}$, we get (see Lemma 3.4)

$$|j - \tilde{j}| = \left|(\rho^2 - \tilde{\rho}^2)(\nabla\varphi - A)\right| \le \left|\frac{\rho^2 - \tilde{\rho}^2}{\rho}\right| |\nabla_A u|.$$

Since from (6.5), $|\rho^2-\tilde\rho^2|/\rho \le 3|1-\rho|$ it follows that $|j-\tilde j| \le 3|1-\rho||\nabla_A u|$ and this has a meaning even if ρ vanishes. From the Cauchy–Schwarz inequality we deduce

$$\|j-\tilde j\|_{L^1(\Omega)} \le 3\|1-\rho\|_{L^2(\Omega)}\|\nabla_A u\|_{L^2(\Omega)} \le 6\sqrt{2}\varepsilon M.$$

Together with (6.7) this proves

$$\left|\int_\Omega \zeta(\mu-\tilde\mu)\right| \le C\varepsilon M\|\nabla\zeta\|_\infty$$

for some universal constant C, hence the lemma. □

6.2 Proof of Theorem 6.1

Throughout the proof C denotes a universal constant. In fact if the constant C was allowed to depend on the domain Ω, the proof would be a bit simplified.

Proof of (6.2)

Using the above lemma, the proof of (6.2) reduces to proving that $\tilde\mu$ defined in (6.6) satisfies

$$\|\tilde\mu-\nu\|_{(C_0^{0,1}(\Omega))^*} \le C\max(r,\varepsilon)(1+M), \tag{6.8}$$

where $\nu = 2\pi\sum_i d_i\delta_{a_i}$ and the sum extends over those balls in $\mathcal{B}$ that are included in Ω_ε. This is equivalent to proving that for any Lipschitz function ζ vanishing on $\partial\Omega$,

$$\left|\int_\Omega \zeta\tilde\mu - \int\zeta\, d\nu\right| \le C(r+\varepsilon)\|\zeta\|_{\text{Lip}}(M+1)\,.$$

The following lemma explains the advantage of working with $\tilde\mu$ rather than μ.

Lemma 6.3. *1. Assume u and A are C^1 and $|u|=1$ identically. Then $\mu=0$, where μ is defined by* (6.1).

2. *Assume u and A are C^1 in a ball B and $|u| = 1$ on the boundary of B. Then*

$$\int_B \mu = 2\pi \deg(u, \partial B).$$

Proof. Since $|u| = 1$, we have $\mu = \operatorname{curl}((iu, \nabla u - iAu) + A) = \operatorname{curl}(iu, \nabla u)$. But writing $u = e^{i\varphi}$ locally for a C^1 function φ, we have $\operatorname{curl}(iu, \nabla u) = \operatorname{curl} \nabla\varphi = 0$. The second assertion is an integration by parts:

$$\int_B \mu = \int_{\partial B} \tau \cdot ((iu, \nabla_A u) + A) = \int_{\partial B} \tau \cdot (iu, \nabla u) = 2\pi \deg(u, \partial B). \qquad \square$$

Now recall that if $|\rho - 1| \leq 1/2$, then $|\tilde{u}| = \chi(\rho) = 1$. Therefore $|\tilde{u}| = 1$ in $\Omega_\varepsilon \setminus \cup_i B_i$. If we let $U = \{x \in \Omega \setminus \Omega_\varepsilon, |\rho(x) - 1| \geq 1/2\}$, then $|\tilde{u}| = 1$ in $(\Omega \setminus \Omega_\varepsilon) \setminus U$. From the above lemma it follows that $\tilde{\mu} = 0$ outside $U \cup (\cup_i B_i)$ and therefore for any Lipschitz function ζ, we have

$$\int_\Omega \zeta\tilde{\mu} = \int_U \zeta\tilde{\mu} + \sum_{i \in I_1} \int_{B_i \cap \Omega} \zeta\tilde{\mu} + \sum_{i \in I_2} \int_{B_i \cap \Omega} \zeta\tilde{\mu},$$

where I_1 is the set of indices i such that $B_i \not\subset \Omega_\varepsilon$ and $I_2 = I \setminus I_1$ is the complement of I_1. If $i \in I_2$, then the previous lemma implies that the integral of $\tilde{\mu}$ over B_i is $2\pi d_i$. Using this and writing $\zeta(x) = \zeta(a_i) + (\zeta(x) - \zeta(a_i))$ in B_i we find

$$\int_\Omega \zeta\tilde{\mu} - 2\pi \sum_{i \in I_2} d_i \zeta(a_i) = \int_U \zeta\tilde{\mu} + \sum_{i \in I_1} \int_{B_i \cap \Omega} \zeta\tilde{\mu} + \sum_{i \in I_2} \int_{B_i \cap \Omega} (\zeta - \zeta(a_i))\tilde{\mu}. \tag{6.9}$$

The left-hand side of this equality is the integral of ζ with respect to the measure $\tilde{\mu}(x)\, dx - \nu$, where ν is defined in (6.8). We need to estimate the right-hand side for any Lipschitz function ζ vanishing on $\partial\Omega$. Our basic tool is the following remark:

Lemma 6.4. *If u and A are C^1 on Ω and μ is defined by* (6.1), *then for any $U \subset \Omega$,*

$$\int_U |\mu| \leq 4(F_\varepsilon(u, A, U) + F_\varepsilon(u, A, U)^{\frac{1}{2}} |U|^{\frac{1}{2}}). \tag{6.10}$$

Proof. It is easy to check that $\mu = 2\partial_x^A u \times \partial_y^A u + h$ and therefore $|\mu| \leq 2|\nabla_A u|^2 + |h|$. Integrating on Ω and using the Cauchy–Schwarz inequality to estimate the integral of $|h|$ yields the result. □

Now we note that any $x \in U$ is also in $\Omega \setminus \Omega_\varepsilon$, hence at a distance less than ε from $\partial\Omega$. Thus, if $\zeta = 0$ on $\partial\Omega$, we find $|\zeta(x)| \leq \varepsilon\|\zeta\|_{\mathrm{Lip}}$. In the same spirit if $i \in I_1$, then B_i intersects the complement of Ω_ε and therefore contains a point at a distance less than ε from $\partial\Omega$. Since the radius of B_i is less than $r(\mathcal{B}) = r$, we find that any $x \in B_i$ satisfies $\operatorname{dist}(x, \partial\Omega) \leq 2r + \varepsilon$ and thus $|\zeta(x)| \leq 2(r+\varepsilon)\|\zeta\|_{\mathrm{Lip}}$. Finally if $i \in I_2$ and $x \in B_i$, then $|\zeta(x) - \zeta(a_i)| \leq r\|\zeta\|_{\mathrm{Lip}}$, since B_i has radius less than r. Inserting this in (6.9) and using the previous lemma we find, letting $V = \cup_{i\in I} B_i$,

$$\left|\int_\Omega \zeta\tilde{\mu} - \int \zeta\, d\nu\right| \leq C(r+\varepsilon)\|\zeta\|_{\mathrm{Lip}}\Big(F_\varepsilon(\tilde{u}, A, U) + F_\varepsilon(\tilde{u}, A, V) + \sqrt{|U|F_\varepsilon(\tilde{u}, A, U)} + \sqrt{|V|F_\varepsilon(\tilde{u}, A, V)}\Big) \quad (6.11)$$

for some universal constant C. Note that from (6.4) and the definition of $\tilde{\mu}$ it follows that $|\nabla_A \tilde{u}| \leq 2|\nabla_A u|$ and $|1 - |\tilde{u}|| \leq |1 - |u||$. This implies that $F_\varepsilon(\tilde{u}, A, \omega) \leq CF_\varepsilon(u, A, \omega)$. Also, since $||u(x)| - 1| \geq 1/2$ for $x \in U$, the integral of $(1 - |u|^2)^2$ on U is greater than $|U|/4$. Therefore $|U| \leq C\varepsilon^2 F_\varepsilon(u, A, U)$. It is clear that $|V| \leq Cr^2$. These remarks show that the right-hand side of (6.11) is bounded above by $C(r+\varepsilon)\|\zeta\|_{\mathrm{Lip}}(F_\varepsilon(u, A, \Omega) + (r+\varepsilon)F_\varepsilon(u, A, \Omega)^{\frac{1}{2}})$ and then, remembering that r and ε are less than 1 and $\sqrt{x} \leq 1 + x$ we get

$$\left|\int_\Omega \zeta\tilde{\mu} - \int \zeta\, d\nu\right| \leq C(r+\varepsilon)\|\zeta\|_{\mathrm{Lip}}\,(M+1)\,.$$

Since this inequality is true for any ζ vanishing on $\partial\Omega$, we have proved (6.2).

Proof of (6.3)

Of course, if the constant in (6.3) was allowed to depend on Ω, this would be a trivial consequence of (6.10), but here additional work is required.

To prove (6.3), we take a continuous bounded function ζ. Then

$$\left|\int_\Omega \zeta\mu\right| \le C\|\zeta\|_\infty \int_\Omega |\mu|.$$

We write $\mu = \tilde\mu + (\mu - \tilde\mu)$. It is easy to check that $|\mu - \tilde\mu| \le C|\nabla_A u|^2$ hence $\|\mu - \tilde\mu\|_{L^1(\Omega)} \le CM$. From (6.10) and Lemma 6.3, we have

$$\int_\Omega |\tilde\mu| = \int_\omega |\tilde\mu| \le 4(F_\varepsilon(\tilde u, A, \omega) + F_\varepsilon(\tilde u, A, \omega)^{\frac12}|\omega|^{\frac12}),$$

where $\omega = \{|\tilde u| \neq 1\} = \{||u| - 1| > 1/2\}$. Arguing as above, we have $|\omega| \le C\varepsilon^2 F_\varepsilon(u, A, \omega)$ and $F_\varepsilon(\tilde u, A, \omega) \le CF_\varepsilon(u, A, \omega)$ therefore

$$\int_\Omega |\tilde\mu| \le C(M + \varepsilon M) \le CM,$$

since we have assumed $\varepsilon \le 1$. It follows that

$$\left|\int_\Omega \zeta\mu\right| \le CM\|\zeta\|_\infty,$$

which proves (6.3).

6.3 A Corollary

Using the very nice interpolation argument of Jerrard–Soner [119] we have:

Theorem 6.2. *Assume $\alpha \in (0,1)$ and $\varepsilon < \varepsilon_0(\alpha)$, where $\varepsilon_0(\alpha)$ is given by Theorem 4.1. Assume $F_\varepsilon(u, A, \Omega) \le \varepsilon^{\alpha-1}$ and let $\mathcal{B} = \{B(a_i, r_i)\}_{i\in I}$ denote a collection of balls given by Theorem 4.1 for some $\varepsilon^{\alpha/2} < r < 1$. We let*

$$\nu = 2\pi \sum_{i\in I/B_i\subset\Omega_\varepsilon} d_i\delta_{a_i}$$

where $d_i = \deg(u, \partial B_i)$, and $\mu = \operatorname{curl}(iu, \nabla_A u) + h$. Then, writing $M = F_\varepsilon(u, A, \Omega)$, we have

$$\|\mu - \nu\|_{C_0^{0,1}(\Omega)^*} \le Cr(M+1) \quad \text{and} \quad \|\nu\|_{C^0(\Omega)^*} \le C\frac{M}{\alpha|\log\varepsilon|},$$

where C is a universal constant. Moreover for any $\beta \in (0,1)$ there exists a constant C_β depending on β and Ω and $\varepsilon_0(\alpha,\beta)$ such that if $\varepsilon < \varepsilon_0$, then

$$\|\mu\|_{C_0^{0,\beta}(\Omega)^*} \le C_\beta \frac{M+1}{\alpha|\log\varepsilon|},$$

and

$$\|\mu - \nu\|_{C_0^{0,\beta}(\Omega)^*} \le C r^\beta (M+1).$$

In particular, if $F_\varepsilon(u, A, \Omega)$ is bounded by $C|\log\varepsilon|$, then μ is bounded in $C_0^{0,\beta}(\Omega)^*$ independently of ε.

The proof relies on the following lemma, taken from [119]:

Lemma 6.5. *Assume ν is a Radon measure on Ω. Then for any $\beta \in (0,1)$,*

$$\|\nu\|_{C_0^{0,\beta}(\Omega)^*} \le \|\nu\|_{C_0^0(\Omega)^*}^{1-\beta} \|\nu\|_{C_0^{0,1}(\Omega)^*}^{\beta}.$$

Proof of the theorem. The fact that $\|\nu\|_{C^0(\Omega)^*} \le CM/(\alpha|\log\varepsilon|)$ follows from Theorem 4.1 (4.4). The bound $\|\mu - \nu\|_{C_0^{0,1}(\Omega)^*} \le Cr(M+1)$ is Theorem 6.1, (6.2).

From now on, C denotes a constant depending possibly on β and Ω. To prove the last assertion we write $\mu = \nu + (\mu - \nu)$. Then

$$\|\mu\|_{C_0^{0,\beta}(\Omega)^*} \le \|\nu\|_{C_0^{0,\beta}(\Omega)^*} + \|\mu - \nu\|_{C_0^{0,\beta}(\Omega)^*}.$$

But

$$\|\nu\|_{C_0^{0,\beta}(\Omega)^*} \le C\|\nu\|_{C_0^0(\Omega)^*} \tag{6.12}$$

and

$$\|\mu - \nu\|_{C_0^{0,\beta}(\Omega)^*} \le \|\mu - \nu\|_{C_0^0(\Omega)^*}^{1-\beta} \|\mu - \nu\|_{C_0^{0,1}(\Omega)^*}^{\beta}. \tag{6.13}$$

But we have already proved that $\|\mu - \nu\|_{C_0^{0,1}(\Omega)^*} \le Cr(M+1)$ and using (6.2) in Theorem 6.1, we have

$$\|\mu - \nu\|_{C_0^0(\Omega)^*} \le \|\mu\|_{C_0^0(\Omega)^*} + \|\nu\|_{C_0^0(\Omega)^*} \le CM + C\frac{M}{\alpha|\log\varepsilon|} \le CM, \tag{6.14}$$

if ε is small enough depending on α. It follows from (6.12), (6.13) and (6.14) that

$$\|\mu - \nu\|_{C_0^{0,\beta}(\Omega)^*} \le C(M+1)r^\beta$$

and

$$\|\mu\|_{C_0^{0,\beta}(\Omega)^*} \le C\frac{M}{\alpha|\log\varepsilon|} + C(M+1)r^\beta.$$

Since μ does not depend on r we may choose $r = \varepsilon^{\alpha/2}$ and then, if ε is small enough depending on α, β we have $r^\beta < 1/(\alpha|\log\varepsilon|)$, hence

$$\|\mu\|_{C_0^{0,\beta}(\Omega)^*} \le C\frac{M}{\alpha|\log\varepsilon|},$$

proving the proposition. □

Bibliographic notes on Chapter 6: The relation between weak Jacobians and the Ginzburg–Landau energy was first emphasized by Jerrard and Soner in [119], where the result was also extended to higher dimensions through a formulation involving currents. However, the method of linking the measures ν to $\mathrm{curl}(iu, \nabla u)$ already appeared in Bethuel–Rivière [51], and also in [52, 181]. A result similar to Theorem 6.1 but with $W^{-1,p}$ estimates instead of $(C^{0,\beta})^*$ was also contained in [168].

Chapter 7

The Obstacle Problem

In this chapter, we start studying the question of minimizing the energy G_ε and we prove the main result of Γ-convergence of G_ε. As already mentioned, configurations have a vorticity $\mu(u_\varepsilon, A_\varepsilon)$, which, according to Chapter 6, is compact as $\varepsilon \to 0$ (under a suitable energy bound) and the result we obtain below shows that minimizers of G_ε have vorticities which converge to a measure which minimizes a certain convex energy. This measure, by convex duality, is shown to be the solution to a simple *obstacle problem*.

The optimal vortex-density and number of vortices will thus be identified as well as the leading order of the energy of minimizers. The Γ-convergence method consists of two steps. First, given a measure μ, we construct a suitable sequence of test-configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ with vorticities converging to μ and which, to leading order as $\varepsilon \to 0$, have the expected optimal energy. Secondly, we obtain a matching lower bound for the energy of configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ with vorticities converging to μ.

We introduce some definitions that will be used throughout the remainder of the book. The function h_0 is the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 \text{ in } \Omega \\ \qquad\qquad h_0 = 1 \text{ on } \partial\Omega, \end{cases} \tag{7.1}$$

and we let

$$\xi_0 = h_0 - 1. \tag{7.2}$$

We also set

$$J_0 = \frac{1}{2} \int_\Omega |\nabla h_0|^2 + |h_0 - 1|^2 = \frac{1}{2} \|\xi_0\|^2_{H^1(\Omega)}. \tag{7.3}$$

Since $0 \le h_0 \le 1$ (by maximum principle), the function ξ_0 is negative in Ω and smooth. We let

$$\underline{\xi_0} = \min_\Omega \xi_0, \tag{7.4}$$

$$\Lambda = \left\{ x \in \Omega \mid \xi_0(x) = \underline{\xi_0} \right\}. \tag{7.5}$$

The following result is proved in [171].

Lemma 7.1. *The set*

$$\{x \in \Omega \mid \nabla \xi_0(x) = 0\}$$

is finite, hence Λ *also.*

We recall that we write

$$j = (iu, \nabla_A u), \quad \mu(u, A) = \operatorname{curl} j + \operatorname{curl} A.$$

We also denote by $\mathcal{M}(\Omega)$ the space of bounded Radon measures on Ω, i.e., $(C^0_0(\Omega))^*$. We denote by $|\mu|$ the total variation of μ, i.e., if $\mu = \mu_+ - \mu_-$ is the canonical representation of μ as the difference of two positive measures, $|\mu| = \mu_+ + \mu_-$. We write $\|\mu\|$ for $|\mu|(\Omega)$.

7.1 Γ-Convergence

In this theorem, $H^1_1(\Omega)$ denotes the affine space of functions of $H^1(\Omega)$ whose trace on the boundary is 1 (or $1 + H^1_0(\Omega)$).

Theorem 7.1 (Γ-convergence of G_ε). *Assume*

$$\frac{h_{ex}}{|\log \varepsilon|} \to \lambda > 0 \quad \text{as } \varepsilon \to 0.$$

Then,

$$\frac{G_\varepsilon}{{h_{ex}}^2} \xrightarrow{\Gamma} E_\lambda(\mu) = \frac{\|\mu\|}{2\lambda} + \frac{1}{2} \int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \tag{7.6}$$

where E_λ *is defined over* $\mathcal{M}(\Omega) \cap H^{-1}(\Omega)$ *and where* h_μ *is the solution of*

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu \text{ in } \Omega \\ \qquad\qquad h_\mu = 1 \text{ on } \partial\Omega. \end{cases} \tag{7.7}$$

More specifically,

1) *If* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *are such that* $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C{h_{ex}}^2$ *and* $\|u_\varepsilon\|_\infty \le 1$ *then, up to extraction, denoting* $j_\varepsilon = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)$ *and* $h_\varepsilon = \operatorname{curl} A_\varepsilon$,

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \longrightarrow \mu \quad \text{in } (C^{0,\gamma}(\Omega))^* \quad \text{as } \varepsilon \to 0$$

for every $\gamma \in (0,1)$ *and*

$$\frac{j_\varepsilon}{h_{ex}} \rightharpoonup j, \qquad \frac{h_\varepsilon}{h_{ex}} \rightharpoonup h, \quad \text{as } \varepsilon \to 0$$

weakly in $L^2(\Omega)$. *Moreover* $\mu = \operatorname{curl} j + h$ *and*

$$\liminf_{\varepsilon \to 0} \frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{ex}}^2} \ge E_\lambda(\mu) + \frac{1}{2} \int_\Omega |j + \nabla^\perp h_\mu|^2 + |h - h_\mu|^2. \tag{7.8}$$

2) *For every* $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, *there exist* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *such that* $\|u_\varepsilon\|_\infty \le 1$,

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \longrightarrow \mu \quad \text{in } (C^{0,\gamma}(\Omega))^* \quad \text{as } \varepsilon \to 0$$

for every $\gamma \in (0,1)$ *and*

$$\frac{h_\varepsilon}{h_{ex}} \to h_\mu \quad \text{as } \varepsilon \to 0$$

weakly in $H^1_1(\Omega)$ *and strongly in* $W^{1,p}(\Omega)$ *for every* $p < 2$, *and such that*

$$\limsup_{\varepsilon \to 0} \frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{ex}}^2} \le E_\lambda(\mu).$$

The functional E_λ defined over $\mathcal{M}(\Omega) \cap H^{-1}(\Omega)$ is strictly convex and continuous, therefore it has a unique minimizer μ_*. From the Γ-convergence result, it is standard to deduce:

Theorem 7.2 (Convergence of minimizers). *Let* $\varepsilon \to 0$ *and* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *be a family of minimizers of* G_ε*, with* $\frac{h_{ex}}{|\log\varepsilon|} \to \lambda > 0$*. Then, as* $\varepsilon \to 0$*,*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \longrightarrow \mu_* \quad \text{in } (C^{0,\gamma}(\Omega))^* \quad \text{for every } \gamma \in (0,1),$$
$$\frac{h_\varepsilon}{h_{ex}} \to h_{\mu_*} \text{ weakly in } H_1^1(\Omega) \text{ and strongly in } W^{1,p}(\Omega),\ \forall p < 2,$$

where μ_* *is the unique minimizer of* E_λ*. Moreover, letting* $g_\varepsilon(u, A)$ *denote the energy density* $\frac{1}{2}\left(|\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$*,*

$$\frac{g_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{ex}}^2} \to \frac{1}{2\lambda}|\mu_*| + \frac{1}{2}\left(|\nabla h_{\mu_*}|^2 + |h_{\mu_*} - 1|^2\right) \tag{7.9}$$

and

$$\left|\nabla\left(\frac{h_\varepsilon}{h_{ex}} - h_{\mu_*}\right)\right|^2 \to \frac{1}{\lambda}\mu_* \tag{7.10}$$

in the weak sense of measures.

The only statements which do not follow directly from Theorem 7.1 are (7.9) and (7.10). They describe the defect measure in the weak convergence of $h_\varepsilon/h_{\text{ex}}$ to h_{μ_*}.

7.2 Description of μ_*

We have the following result:

Proposition 7.1 (Dual problem, see [59, 64]). *Given a continuous function* $p \geq 0$*, the minimizer of*

$$\min_{\substack{u \in H_0^1(\Omega) \\ -\Delta u + u \in \mathcal{M}(\Omega)}} \lambda\|-\Delta u + u + p\| + \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 \tag{7.11}$$

is the minimizer of the dual problem

$$\min_{\substack{v \in H_0^1(\Omega) \\ |v| \leq \lambda}} \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + v^2 + 2pv\right), \tag{7.12}$$

or equivalently if p *is constant,* $u+p$ *is the minimizer of*

$$\min_{\substack{f\in H^1_p(\Omega)\\ f\ge p-\lambda}} \frac{1}{2}\int_\Omega |\nabla f|^2 + f^2.$$

It always satisfies $-\Delta u + u + p \ge 0$.

Here $H^1_p(\Omega)$ denotes the affine space $p + H^1_0(\Omega)$.

Proof. The result relies on the following result of convex duality (see [91] for example):

Lemma 7.2. *Let* Φ *be convex lower semi-continuous from a Hilbert space* H *to* $(-\infty,+\infty]$*, and let* Φ^* *denote its conjugate, i.e.,*

$$\Phi^*(f) = \sup_{g/\Phi(g)<\infty} \langle f, g\rangle_H - \Phi(g), \tag{7.13}$$

then

$$\min_{u\in H}\left(\frac{1}{2}\|u\|_H^2 + \Phi(u)\right) = -\min_{h\in H}\left(\frac{1}{2}\|h\|_H^2 + \Phi^*(-h)\right)$$

and minimizers coincide.

Now we apply this to $H = H^1_0(\Omega)$ with the norm $\|h\|_H^2 = \int_\Omega |\nabla h|^2 + h^2$, and $\Phi(u) = \lambda\| - \Delta u + u + p\|_{\mathcal{M}(\Omega)}$ defined over the set of $u \in H^1_0$ such that $-\Delta u + u \in \mathcal{M}(\Omega)$. Using the definition (7.13), we find

$$\Phi^*(f) = \begin{cases} -\displaystyle\int_\Omega pf & \text{if } |f|\le\lambda \text{ a.e.} \\ +\infty & \text{otherwise.}\end{cases}$$

Indeed,

$$\sup_{\substack{g\in H^1_0(\Omega)\\ -\Delta g+g\in\mathcal{M}(\Omega)}} \int_\Omega \nabla g\cdot\nabla f + gf - \lambda\| - \Delta g + g + p\| =$$

$$\sup_{\substack{g\in H^1_0(\Omega)\\ -\Delta g+g\in\mathcal{M}(\Omega)}} \int_\Omega f\, d(-\Delta g + g + p) - \lambda\| - \Delta g + g + p\| - \int_\Omega pf$$

$$\ge \sup_{\zeta\in L^2}\int_\Omega f\zeta - \lambda\int_\Omega|\zeta| - \int_\Omega pf.$$

We deduce $\Phi^*(f) = +\infty$ if $|f| \leq \lambda$ is not satisfied a.e. and in any case $\Phi^*(f) \geq -\int_\Omega pf$ (take $\zeta = 0$). If $|f| \leq \lambda$ the converse inequality $\Phi^*(f) \leq -\int_\Omega pf$ is clear.

Thus, applying the lemma, we find that the minimizer of (7.11) is the minimizer of (7.12). The remaining assertions are easy consequences of (7.12) and the maximum principle. □

Applying Proposition 7.1 with $u = h - 1$ and $p = 1$, we deduce the following:

Corollary 7.1. *The function h_{μ_*} introduced in Theorem 7.2 is also the unique minimizer of the following* obstacle problem

$$\min_{\substack{h \geq 1 - \frac{1}{2\lambda} \\ h \in H_1^1(\Omega)}} \frac{1}{2} \int_\Omega |\nabla h|^2 + h^2. \tag{7.14}$$

It is characterized by the fact that $h_{\mu_*} \in H_1^1(\Omega)$, and that $h_{\mu_*} \geq 1 - 1/(2\lambda)$ in Ω and the following variational inequality (see [126] for references on such variational inequalities)

$$\int_\Omega (-\Delta h_{\mu_*} + h_{\mu_*})(v - h_{\mu_*}) \geq 0$$

for every $v \in H_1^1(\Omega)$ such that $v \geq 1 - 1/(2\lambda)$.

From this we deduce, in particular, that

$$\mu_* = -\Delta h_{\mu_*} + h_{\mu_*} \geq 0,$$

i.e., the limiting measure is positive.

The obstacle problem (7.14) is well studied (see [126] for further reference), in particular the regularity of solutions is well understood. The function h_{μ_*} belongs to $C^{1,\alpha}(\Omega)$ for every $\alpha < 1$ (see [99]). The measure μ_* can be described in terms of the *coincidence set*

$$\omega_\lambda = \left\{ x \in \Omega \mid h_{\mu_*}(x) = 1 - \frac{1}{2\lambda} \right\}$$

by the following relation, where $\mathbf{1}_{\omega_\lambda}\, dx$ denotes the Lebesgue measure restricted to ω_λ:

$$\boxed{\mu_* = \left(1 - \frac{1}{2\lambda}\right) \mathbf{1}_{\omega_\lambda}\, dx}. \tag{7.15}$$

This follows from the fact that where $h_{\mu_*} > 1 - 1/(2\lambda)$, the function h_{μ_*} satisfies the unconstrained Euler–Lagrange equations for the problem, i.e., $-\Delta h_{\mu_*} + h_{\mu_*} = 0$. We thus see that μ_* is constant on its support, i.e., there is a uniform limiting density of vortices in ω_λ, a first step towards the Abrikosov lattice.

The regularity of the free boundary $\partial\omega_\lambda$ is a delicate question, however in dimension 2 there is a rather complete theory. It is known (see [65]) that the free boundary is analytic except at a finite number of singular points and that ([144, 56]) for almost every λ there are no singular points at all. For further results we refer to the survey [145]. Note that if $\partial\omega_\lambda$ is smooth then h_{μ_*} can be characterized as the solution of the over-determined system

$$\begin{cases} -\Delta h_{\mu_*} + h_{\mu_*} = 0 & \text{in } \Omega\backslash\omega_\lambda \\ h_{\mu_*} = 1 - \dfrac{1}{2\lambda} & \text{in } \omega_\lambda \\ \dfrac{\partial h_{\mu_*}}{\partial\nu} = 0 & \text{on } \partial\omega_\lambda \\ h_{\mu_*} = 1 & \text{on } \partial\Omega. \end{cases}$$

Such a system was derived by Chapman, Rubinstein and Schatzman in [72] by formal arguments, and we may see Theorem 7.2 as giving a rigorous derivation of it from the minimization of the Ginzburg–Landau functional.

Below, we collect some facts about ω_λ, h_*, μ_*, whose proofs rely entirely on the maximum principle.

Proposition 7.2. *We have the following.*

1. *ω_λ is increasing with respect to λ and $\cup_{\lambda>0}\omega_\lambda = \Omega$. Moreover $\Omega\backslash\omega_\lambda$ is connected.*

2. *For $\lambda < \frac{1}{2|\underline{\xi_0}|}$ we have $h_{\mu_*} = h_0$, $\mu_* = 0$, $\omega_\lambda = \varnothing$.*

3. *For $\lambda = \frac{1}{2|\underline{\xi_0}|}$ we have $h_{\mu_*} = h_0$, $\mu_* = 0$, $\omega_\lambda = \Lambda$.*

4. *For $\lambda > \frac{1}{2|\underline{\xi_0}|}$ we have $\mu_* \neq 0$, ω_λ strictly contains Λ, and (7.15) holds.*

The above motivates the introduction of the following notation

$$\boxed{H_{c_1}^0 = \frac{1}{2|\underline{\xi_0}|}|\log\varepsilon|}. \tag{7.16}$$

In view of Proposition 7.2, the value $H_{c_1}^0$ appears critical in the sense that below $H_{c_1}^0$, the limiting vortex-density μ_* for energy-minimizers (after rescaling by $h_{\rm ex}$) is 0. Above $H_{c_1}^0$, the limiting vortex-density (after rescaling by $h_{\rm ex}$) is nonzero, it has a uniform density $1 - \frac{1}{2\lambda} > 0$ in the subdomain ω_λ of Ω, that is there should be vortices uniformly scattered in ω_λ — hence a number of vortices proportional to $h_{\rm ex}$, itself proportional to $|\log\varepsilon|$ — and a peripheral region without vortices (see again Fig. 1.3).

The usual notion of first critical field is more like the following, though. For a fixed value of ε, it is the value $H_{c_1}(\varepsilon)$ such that if $h_{\rm ex} < H_{c_1}(\varepsilon)$, then there exists a minimizer (u, A) of G_ε such that $|u| > 0$ in Ω while if $h_{\rm ex} > H_{c_1}(\varepsilon)$ and (u, A) is a minimizer, then u must vanish in Ω. A priori H_{c_1} and $H_{c_1}^0$ could be very different numbers. This would be the case if for a wide range of $h_{\rm ex}$ minimizers of G_ε had vortices, but few of them compared to $h_{\rm ex}$, because the rescaled limiting measure μ would still be zero. We will prove in Chapter 12 that this cannot happen and that $H_{c_1}^0$ is the leading order of H_{c_1} as $\varepsilon \to 0$, confirming the physics knowledge.

We now present the full proof of Theorem 7.1.

7.3 Upper Bound

In this section we will prove item 2) in Theorem 7.1, which we state as Proposition 7.5 below. However, the intermediate results will also be useful in subsequent chapters.

First, we show that given a set $\{(a_i, d_i)\}_i$ of points and degrees, and $\varepsilon > 0$, we may associate to it a configuration (u, A) having $\{(a_i, d_i)\}_i$ as vortices, and express $G_\varepsilon(u, A)$ as a function of ε and $\{(a_i, d_i)\}_i$. Second we show that using the above with a well chosen family $\{(a_i, d_i)\}_i$ yields the desired upper bound.

7.3.1 The Space H^1 and the Green Potential

We introduce the (modified) Green's function G_Ω associated to a smooth bounded domain Ω in $\mathbb{R}^2$, as the solution of

$$\begin{cases} -\Delta_x G_\Omega(x,y) + G_\Omega(x,y) = \delta_y & \text{in } \Omega \\ G_\Omega(x,y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.17}$$

and let

$$S_\Omega(x,y) = 2\pi G_\Omega(x,y) + \log|x-y|. \tag{7.18}$$

It is a standard fact that G_Ω is symmetric, positive, and the function S_Ω is C^1 in $\Omega\times\Omega$. Note however that S_Ω is not bounded up to the boundary.

In addition to (7.7), if $\mu \in H^{-1}(\Omega)$, we introduce the notation U_μ for the solution of

$$\begin{cases} -\Delta U_\mu + U_\mu = \mu \text{ in } \Omega \\ U_\mu = 0 \text{ on } \partial\Omega. \end{cases} \tag{7.19}$$

In addition, when μ is a bounded Radon measure we have

$$U_\mu(x) = \int G_\Omega(x,y)\, d\mu(y).$$

Indeed Fubini's theorem shows that the integral defines an L^1 function for any measure μ, and it can be checked that when $\mu \in H^{-1}$ this function is the solution of (7.19).

The following property is also true (this follows from [60], Theorem 1). If μ, ν are *positive* Radon measures in $H^{-1}(\Omega)$, then $U_\mu \in L^1(d\nu)$ and

$$\langle U_\mu, U_\nu\rangle_{H^1(\Omega)} = \iint G_\Omega(x,y)\, d\mu(x)\, d\nu(y). \tag{7.20}$$

Thus, if $\mu = \mu_+ - \mu_-$ is a signed measure with μ_+ and μ_- positive measures belonging to $H^{-1}(\Omega)$, writing the above relations for the couples (μ_+,μ_+), (μ_-,μ_-), (μ_+,μ_-) and combining them, we find that

$$\int_\Omega |\nabla U_\mu|^2 + {U_\mu}^2 = \iint G_\Omega(x,y)\, d\mu(x)\, d\mu(y). \tag{7.21}$$

7.3.2 The Energy-Splitting Lemma

Here we describe an elementary way of splitting the energy relating G_ε and F_ε, which we will use many times in the sequel, in particular in Chapters 9, 10, and 11. It was first observed by Bethuel and Rivière in [51].

Lemma 7.3 (Energy-splitting). *For any* (u, A)*, denoting* $A' = A - h_{ex}\nabla^\perp h_0$*, we have*

$$G_\varepsilon(u, A) = {h_{ex}}^2 J_0 + F_\varepsilon(u, A') + h_{ex} \int_\Omega \xi_0 \mu(u, A') + R_0, \tag{7.22}$$

where J_0 *was defined in (7.3),* ξ_0 *in (7.2),* F_ε *in (4.1), and*

$$R_0 = \frac{1}{2} \int_\Omega {h_{ex}}^2 (|u|^2 - 1)|\nabla h_0|^2 \le C\varepsilon {h_{ex}}^2 F_\varepsilon(|u|)^{\frac{1}{2}}.$$

Proof. Let us write A as $h_{\rm ex}\nabla^\perp h_0 + A'$, plug it into $G_\varepsilon(u, A)$ and expand the square terms. This easily yields, using the fact that $\operatorname{curl}\nabla^\perp h_0 = \Delta h_0 = h_0$ (from (7.1)),

$$G_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_{A'} u|^2 + {h_{\rm ex}}^2 |u|^2 |\nabla h_0|^2 - 2h_{\rm ex}(\nabla_{A'} u, iu) \cdot \nabla^\perp h_0$$
$$+ \frac{1}{2} \int_\Omega |\operatorname{curl} A'|^2 + {h_{\rm ex}}^2 |h_0 - 1|^2 + 2h_{\rm ex}(h_0 - 1)\operatorname{curl} A' + \frac{(1 - |u|^2)^2}{2\varepsilon^2}.$$

Therefore, grouping terms, writing $|u|^2$ as $1 + (|u|^2 - 1)$, and integrating by parts, we find

$$G_\varepsilon(u, A) = F_\varepsilon(u, A') + {h_{\rm ex}}^2 J_0 + \frac{1}{2} \int_\Omega {h_{\rm ex}}^2 (|u|^2 - 1)|\nabla h_0|^2$$
$$+ \int_\Omega h_{\rm ex}(h_0 - 1)\operatorname{curl}(iu, \nabla_{A'} u) + h_{\rm ex}(h_0 - 1)\operatorname{curl} A'.$$

This is the result, the upper bound for R_0 following from the Cauchy–Schwarz inequality applied to the integral of $(1 - |u|^2)$. □

7.3.3 Configurations with Prescribed Vortices

Proposition 7.3. *Given $\varepsilon \in (0,1)$ and a set of n points $a_i \in \Omega$ and degrees $d_i = \pm 1$ such that $|a_i - a_j| \geq 8c\varepsilon$ for $i \neq j$ and $\operatorname{dist}(a_i, \partial\Omega) \geq 8c$ for some $c > 0$, there exists a configuration (u, A) such that, μ_i being the uniform measure on $\partial B(a_i, c\varepsilon)$ of mass $2\pi d_i$, and letting*

$$\mu = \sum_{i=1}^{n} \mu_i,$$

we have

$$F_\varepsilon(u, A') = \pi n |\log \varepsilon| + \frac{1}{2} \sum_{i \neq j} \iint G_\Omega(x, y)\, d\mu_i(x)\, d\mu_j(y) + O(n), \tag{7.23}$$

$$G_\varepsilon(u, A) = F_\varepsilon(u, A') + {h_{ex}}^2 J_0 + h_{ex} \int \xi_0(x)\, d\mu(x) + O\left(n\varepsilon h_{ex} + n\varepsilon^2 {h_{ex}}^2 + (n^{1/2}\varepsilon h_{ex} + \varepsilon {h_{ex}}^2) F_\varepsilon(u, A')^{1/2}\right), \tag{7.24}$$

and

$$\frac{1}{n} \|\mu(u, A) - \mu\|_{(C^{0,\gamma}(\Omega))^*} \leq C\varepsilon^\gamma \left(1 + \varepsilon h_{ex} + \left(\frac{F_\varepsilon(u, A')}{n}\right)^{\frac{1}{2}}\right) \quad \forall\ 0 < \gamma \leq 1. \tag{7.25}$$

Here $A' = A - h_{ex} \nabla^\perp h_0$ and the O's depend on Ω and c only.

The proof is in four steps.

Step 1: Construction of the test-configuration.

Let μ_i denote the uniform measure on $\partial B(a_i, c\varepsilon)$ of mass $2\pi d_i$. We define h to be the solution of

$$\begin{cases} -\Delta h + h = \mu = \sum_{i=1}^{n} \mu_i & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega, \end{cases} \tag{7.26}$$

Then, we let A be any vector field such that $\operatorname{curl} A = h$ in Ω and we define $u = \rho e^{i\varphi}$ as follows. First we let

$$\rho(x) = \begin{cases} 0 & \text{if } |x-a_i| \le c\varepsilon \text{ for some } i, \\ \dfrac{|x-a_i|}{c\varepsilon} - 1 & \text{if } c\varepsilon < |x-a_i| < 2c\varepsilon \text{ for some } i, \\ 1 & \text{otherwise,} \end{cases} \tag{7.27}$$

and for any $x \in \widetilde{\Omega}_\varepsilon = \Omega \setminus \cup_i B(a_i, c\varepsilon)$,

$$\varphi(x) = \oint_{(x_0,x)} (A - \nabla^\perp h) \cdot \tau \, d\ell, \tag{7.28}$$

where x_0 is a base point in $\widetilde{\Omega}_\varepsilon$, (x_0, x) is any curve joining x_0 to x in $\widetilde{\Omega}_\varepsilon$, and τ is the unit tangent vector to the curve. From (7.26), we see that this definition of $\varphi(x)$ does not depend modulo 2π on the particular curve (x_0, x) chosen, hence $e^{i\varphi}$ is well-defined. Indeed, if $\gamma = \partial U$ is a boundary in $\widetilde{\Omega}_\varepsilon$, then using Stokes's theorem and $\operatorname{curl} A = h$, we find

$$\int_\gamma (A - \nabla^\perp h) \cdot \tau \, d\ell = \int_U -\Delta h + h = 2\pi \sum_{a_i \in U} d_i \in 2\pi\mathbb{Z},$$

since γ does not intersect the $\overline{B}(a_i, c\varepsilon)$'s. From (7.28), the function φ satisfies

$$-\nabla^\perp h = \nabla\varphi - A \tag{7.29}$$

in $\widetilde{\Omega}_\varepsilon$. Finally, we define

$$u = \rho e^{i\varphi}.$$

Observe that the fact that φ is not defined on $\cup_i B(a_i, c\varepsilon)$ is not important since ρ is zero there.

Step 2: Free energy of (u, A).

Having defined (u, A) as above, we estimate $F_\varepsilon(u, A')$. Recall that

$$F_\varepsilon(u, A') = \frac{1}{2} \int_\Omega |\nabla\rho|^2 + \rho^2 |\nabla\varphi - A'|^2 + |h|^2 + \frac{1}{2\varepsilon^2} \left(1 - \rho^2\right)^2. \tag{7.30}$$

From (7.27) we have

$$\frac{1}{2}\int_{B(a_i,2c\varepsilon)} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2 \le C.$$

Then, using the fact that the number of points is n and that $\rho = 1$ in $\Omega \setminus \cup_i B(a_i, 2c\varepsilon)$, it follows that

$$\frac{1}{2}\int_{\Omega} |\nabla\rho|^2 + \frac{1}{2\varepsilon^2}\left(1-\rho^2\right)^2 \le Cn. \tag{7.31}$$

Also, from (7.27)–(7.29) and the facts that $A' = A - h_{\mathrm{ex}}\nabla^\perp h_0$ and $\Delta h_0 = h_0$, we have

$$\rho^2|\nabla\varphi - A'|^2 \le |\nabla\varphi - A'|^2 = |\nabla(h - h_{\mathrm{ex}}h_0)|^2$$

in $\widetilde{\Omega}_\varepsilon$. Therefore, replacing this in (7.30) and in view of (7.31) we find

$$F_\varepsilon(u, A') \le \frac{1}{2}\int_{\Omega} |\nabla(h - h_{\mathrm{ex}}h_0)|^2 + |h - h_{\mathrm{ex}}h_0|^2 + O(n). \tag{7.32}$$

Because h is the solution of (7.26), referring to (7.19) and (7.1), we have $h - h_{\mathrm{ex}}h_0 = U_\mu$. Thus, using (7.21), the inequality (7.32) becomes

$$F_\varepsilon(u, A') \le \frac{1}{2}\iint G_\Omega(x,y)\, d\mu(x)\, d\mu(y) + O(n). \tag{7.33}$$

We now decompose the double integral by writing $\mu = \sum_i \mu_i$ to find,

$$\iint G_\Omega(x,y)\, d\mu(x)\, d\mu(y) = \sum_{i=1}^{n}\iint G_\Omega(x,y)\, d\mu_i(x)\, d\mu_i(y) + \\ + \sum_{i\neq j} d_i d_j \iint G_\Omega(x,y)\, d\mu_i(x)\, d\mu_j(y). \tag{7.34}$$

We now check that

$$\sum_{i=1}^{n}\iint G_\Omega(x,y)\, d\mu_i(x)\, d\mu_i(y) = 2\pi n|\log\varepsilon| + O(n). \tag{7.35}$$

To prove this, recall that $S_\Omega = 2\pi G_\Omega(x,y) + \log|x-y|$ is C^1 in $\Omega\times\Omega$, hence it is locally bounded. Thus there exists a constant C depending only on Ω and c such that $|S_\Omega(x,y)| \le C$ for any $x, y \in \cup_i B(a_i, c\varepsilon)$ since the points a_i are at a distance at least $8c$ from the boundary. It follows that for any $(x,y) \in \operatorname{supp} \mu_i \times \operatorname{supp} \mu_i$ we have $|2\pi G_\Omega(x,y) - \log|x-y|| \le C$ and then, since

$$\iint \log|x-y| d\mu_i(x)\, d\mu_i(y) = \iint_{[0,2\pi]^2} \log|c\varepsilon e^{i\theta} - c\varepsilon e^{i\phi}|\, d\theta\, d\phi = 4\pi^2 \log\varepsilon + C,$$

we have proved (7.35). In view of (7.33)–(7.34), we have constructed a configuration such that

$$F_\varepsilon(u, A') \le \pi n |\log\varepsilon| + \frac{1}{2} \sum_{i \ne j} \iint G_\Omega(x,y)\, d\mu_i(x)\, d\mu_j(y) + O(n).$$

In order to find the desired configuration, if the inequality above is not an equality, we just need to "add" some energy to (u, A'). In order to do so, we keep the same ρ but modify φ outside of the $B(a_i, 2c\varepsilon)$, adding oscillations to it in such a way that $\int |\nabla\varphi - A'|^2$ becomes large enough and reaches the desired value.

Step 3: Proof of (7.25).

Let us still denote by φ the phase that was constructed in Step 1, and let us denote by ψ the oscillations that were possibly added in the end of Step 2 (recall ψ is compactly supported in $\Omega \backslash \cup_i B(a_i, 2c\varepsilon)$). By definition, $\mu(u, A) = \operatorname{curl}((iu, \nabla_A u) + A)$, and thus from (7.29) for example

$$\mu(u, A) = \operatorname{curl}((\nabla(\varphi + \psi) - A) + A) = \Delta h + h = 0 \quad \text{in } \Omega \backslash \cup_i B(a_i, 2c\varepsilon).$$

Thus, $\mu(u, A)$ and $\sum_{i=1}^n \mu_i$ are both zero in $\Omega \backslash \cup_i B(a_i, 2c\varepsilon)$, moreover they have the same mass in each $B(a_i, 2c\varepsilon)$ since

$$\begin{aligned} \int_{B(a_i,2c\varepsilon)} \mu(u, A) &= \int_{\partial B(a_i,2c\varepsilon)} \frac{\partial\varphi}{\partial\tau} = \int_{\partial B(a_i,2c\varepsilon)} \frac{\partial\varphi}{\partial\tau} - A \cdot \tau + \int_{B(a_i,2c\varepsilon)} h \\ &= \int_{B(a_i,2c\varepsilon)} -\Delta h + h = \mu_i(B(a_i, 2c\varepsilon)). \end{aligned}$$

Letting ξ be a smooth compactly supported test-function, we have

$$\begin{aligned}\int_\Omega \left(\mu(u,A) - \sum_{i=1}^n \mu_i\right)\xi &= \sum_{i=1}^n \int_{B(a_i,2c\varepsilon)} (\mu(u,A)-\mu_i)\xi \\ &= \sum_{i=1}^n \int_{B(a_i,2c\varepsilon)} (\mu(u,A)-\mu_i)(\xi - \xi(a_i)).\end{aligned} \tag{7.36}$$

But, recalling that

$$\begin{aligned}\mu(u,A) &= \operatorname{curl}(iu,\nabla_A u) + \operatorname{curl} A \\ &= \nabla^\perp \rho^2 \cdot (\nabla\varphi - A) + (1-\rho^2)h,\end{aligned} \tag{7.37}$$

we have

$$\begin{aligned}&\sum_{i=1}^n \int_{B(a_i,2c\varepsilon)} |\mu(u,A)-\mu_i| \\ &\qquad \le 2\pi n + \int_{\cup_i B(a_i,2c\varepsilon)} |\nabla\rho||\nabla h| + |h| \\ &\qquad \le Cn + \int_{\cup_i B(a_i,2c\varepsilon)} \left(\frac{C}{\varepsilon}|\nabla h'| + \frac{Ch_{\mathrm{ex}}}{\varepsilon} + |h'| + Ch_{\mathrm{ex}}\right)\end{aligned}$$

where we used the fact that $|\nabla\rho| \le \frac{1}{\varepsilon}$. Using the Cauchy–Schwarz inequality, we find

$$\sum_{i=1}^n \int_{B(a_i,2c\varepsilon)} |\mu(u,A)-\mu_i| \le Cn + Cn\varepsilon h_{\mathrm{ex}} + \sqrt{nF_\varepsilon(u,A')}.$$

Combining this with (7.36) and using the fact that $\xi \in C_0^{0,\gamma}(\Omega)$, we conclude that

$$\left|\frac{1}{n}\int_\Omega \left(\mu(u,A) - \sum_{i=1}^n \mu_i\right)\xi\right| \le C\varepsilon^\gamma \left(1 + \varepsilon h_{\mathrm{ex}} + \sqrt{\frac{F_\varepsilon(u,A')}{n}}\right) \|\xi\|_{C^{0,\gamma}(\Omega)}$$

i.e., (7.25) holds.

Step 4: Proof of (7.24).

The relation follows from the energy-splitting lemma, Lemma 7.3, which yields

$$G_\varepsilon(u,A) = {h_{\rm ex}}^2 J_0 + F_\varepsilon(u,A') + h_{\rm ex}\int_\Omega \xi_0\mu(u,A') + O\left(\varepsilon {h_{\rm ex}}^2 F_\varepsilon(|u|)^{\frac12}\right).$$

In order to conclude, we essentially need to estimate $\mu(u,A')-\mu$. First we recall (see (7.37)) that $\mu(u,A) = \mathrm{curl}(\rho^2(\nabla\varphi - A)+A)$ and similarly $\mu(u,A') = \mathrm{curl}(\rho^2(\nabla\varphi - A')+A')$. Thus $\mu(u,A)-\mu(u,A') = \mathrm{curl}\left((1-\rho^2)(A-A')\right) = \mathrm{curl}\left((1-\rho^2)h_{\rm ex}\nabla^\perp h_0\right)$. We deduce, after integration by parts, that

$$\begin{aligned} h_{\rm ex}\int_\Omega \xi_0(\mu(u,A')-\mu(u,A)) &= \int_\Omega {h_{\rm ex}}^2(1-\rho^2)|\nabla h_0|^2 \\ &= O(\varepsilon {h_{\rm ex}}^2 F_\varepsilon(|u|)^{1/2}). \end{aligned} \tag{7.38}$$

On the other hand, from (7.25) and the fact that $\xi_0\in C_0^{0,1}(\Omega)$, we find that

$$\left| h_{\rm ex}\int_\Omega \xi_0(\mu(u,A)-\mu)\right| \le n h_{\rm ex}\varepsilon\left(1+\varepsilon h_{\rm ex} + \sqrt{\frac{F_\varepsilon(u,A')}{n}}\right).$$

Combining this with (7.38) we conclude that (7.24) holds and Proposition 7.3 is proved.

Remark 7.1. *With the same arguments as in this proof, we could easily prove another estimate on the energy of the configuration constructed in Step 1: namely that*

$$\begin{aligned} G_\varepsilon(u,A) \le \pi n|\log\varepsilon| &+ \frac12\sum_{i\neq j}\iint G_\Omega(x,y)\,d\mu_i(x)\,d\mu_j(y) \\ &+ {h_{ex}}^2 J_0 + h_{ex}\int \xi_0(x)d\mu(x) + O(n). \end{aligned}$$

7.3.4 Choice of the Vortex Configuration

Proposition 7.4. *Assume that μ is continuous, compactly supported in Ω and different from 0. Assume that $\{n(\varepsilon)\}_{\varepsilon>0}$ are integers such that $1\ll n\le \frac{C}{\varepsilon^2}$ as $\varepsilon\to 0$.*

Then, there exists $c > 0$ and for every $\varepsilon \in (0,1)$ a family of points $a_i^\varepsilon \in \Omega$ and degrees $d_i^\varepsilon = \pm 1$ such that $|a_i^\varepsilon - a_j^\varepsilon| > 8c\varepsilon$ for every $i \neq j$, $\mathrm{dist}(a_i^\varepsilon, \partial\Omega) > 8c$ *and such that*

$$\frac{1}{n}\sum_{i=1}^{n} \mu_i^\varepsilon \rightharpoonup 2\pi \frac{\mu}{\|\mu\|} \quad \textit{in the weak sense of measures,}$$

for μ_i^ε any measure supported in $\overline{B}(a_i^\varepsilon, c\varepsilon)$, of constant sign, and such that $\mu_i^\varepsilon(\Omega) = 2\pi d_i$. Moreover,

$$\lim_{\varepsilon\to 0} -\frac{1}{n^2}\sum_{i\neq j} \iint \log|x-y| d\mu_i^\varepsilon(x)\, d\mu_j^\varepsilon(y) \tag{7.39}$$
$$= -\frac{4\pi^2}{\|\mu\|^2}\iint \log|x-y|\, d\mu(x)\, d\mu(y).$$

$$\lim_{\varepsilon\to 0} \frac{1}{n^2}\sum_{i\neq j} \iint G_\Omega(x,y) d\mu_i^\varepsilon(x)\, d\mu_j^\varepsilon(y) \tag{7.40}$$
$$= \frac{4\pi^2}{\|\mu\|^2}\iint G_\Omega(x,y)\, d\mu(x)\, d\mu(y).$$

Let $M = \|\mu\|_{L^\infty(\Omega)}$. Let us partition Ω into squares K of sidelength δ where $\delta(\varepsilon)$ is chosen such that

$$\frac{1}{\sqrt{n}} \ll \delta(\varepsilon) \ll 1. \tag{7.41}$$

(Recall that n depends on ε and $n(\varepsilon) \gg 1$.) Let us denote by $\mathcal{K}(\varepsilon)$ the family of those squares that lie entirely inside Ω. The next step is to determine how many points to put in each square.

Lemma 7.4. *Given $n \in \mathbb{N}$ and nonnegative numbers $(\lambda_i)_{1\le i\le \ell}$ with $\sum_{i=1}^{\ell} \lambda_i = n$, we can find integers $(m_i)_{1\le i\le\ell}$ such that*

$$\sum_{i=1}^{\ell} m_i = n,$$
$$|m_i - \lambda_i| < 1 \quad \forall i \in [1,\ell].$$

Proof. We let $[x]$ denote the largest integer less than or equal to x. Letting $\sigma_i = \lambda_1 + \cdots + \lambda_i$ if $i \ge 1$ and $\sigma_0 = 0$, we let $s_i = [\sigma_i]$ for $0 \le i \le \ell$ and define $m_i = s_i - s_{i-1}$, for $1 \le i \le \ell$. Then $m_1 + \cdots + m_\ell = [\sigma_\ell] = n$, and it follows from the inequalities

$$\sigma_i - 1 < s_i \le \sigma_i, \quad \sigma_{i-1} - 1 < s_{i-1} \le \sigma_{i-1}$$

that $\lambda_i - 1 < m_i < \lambda_i + 1$. ∎

We may apply this lemma to the family of real numbers $\{\lambda_K\}_{K\in\mathcal{K}(\varepsilon)}$ defined by

$$\lambda_K = n\frac{|\mu(K)|}{M_\varepsilon},$$

where M_ε is the sum of the numbers $|\mu(K)|$ for $K \in \mathcal{K}(\varepsilon)$. Hence the sum of the λ_K's is n. Note that since μ is continuous with compact support in Ω and since the sidelength δ tends to 0 as $\varepsilon \to 0$, we have

$$M_\varepsilon = \sum_{K\in\mathcal{K}(\varepsilon)} |\mu(K)| \longrightarrow \|\mu\| \quad \text{as } \varepsilon \to 0. \tag{7.42}$$

We deduce from Lemma 7.4 that there exist integers $m_\varepsilon(K)$ such that

$$\sum_{K\in\mathcal{K}(\varepsilon)} m_\varepsilon(K) = n \tag{7.43}$$

and

$$\left| m_\varepsilon(K) - n\frac{|\mu(K)|}{M_\varepsilon} \right| < 1. \tag{7.44}$$

Since $\|\mu\|_\infty \le M$ we have $|\mu|(K) \le M\delta^2$ and therefore $m_\varepsilon(K) \le 1 + nM\delta^2/\|\mu\| = O(nM\delta^2)$. Thus we may pick $m_\varepsilon(K)$ points a_i^ε evenly scattered in K such that

$$|a_i^\varepsilon - a_j^\varepsilon| \ge \frac{C\delta}{\sqrt{m_\varepsilon(K)}} \ge \frac{C}{\sqrt{n}}. \tag{7.45}$$

Therefore, from the hypothesis on n, there exists $c > 0$ such that

$$|a_i^\varepsilon - a_j^\varepsilon| \ge 8c\varepsilon.$$

Moreover, since μ is compactly supported in Ω and making c smaller if necessary, we may assume that the support of μ is at a distance greater than $16c$ from $\partial\Omega$. Then, for ε small enough, we will have, again using the fact that δ goes to zero

$$\operatorname{dist}(a_i^\varepsilon, \partial\Omega) \ge 16c - \sqrt{2}\delta \ge 8c,$$

as required for Proposition 7.3 to apply.

If $\mu(K) \ge 0$ we assign the degree $d_i = 1$ to each $a_i^\varepsilon \in K$, otherwise the degree $d_i = -1$.

Claim: Let μ_i^ε denote a measure of constant sign supported in $\overline{B}(a_i^\varepsilon, c\varepsilon)$ and such that $\mu_i^\varepsilon(\Omega) = 2\pi d_i$. If $i \neq j$, then μ_i^ε and μ_j^ε have disjoint supports and we have

$$\mu_\varepsilon := \frac{1}{n}\sum_i \mu_i^\varepsilon \rightharpoonup 2\pi\frac{\mu}{\|\mu\|} \tag{7.46}$$

in the sense of measures.

Proof. From (7.42), it suffices to prove that

$$\mu_\varepsilon - \nu_\varepsilon \rightharpoonup 0 \tag{7.47}$$

in the sense of measures, where

$$\nu_\varepsilon = 2\pi\frac{\mu}{M_\varepsilon}.$$

Let f be in $C_0^0(\Omega)$. For any $K \in \mathcal{K}_\varepsilon$, we may decompose f as $(f - \bar{f}) + \bar{f}$, where $\bar{f}$ is the average of f on K with respect to ν_ε, which yields

$$\int_K f\, d\,(\nu_\varepsilon - \mu_\varepsilon) = -\int_K (f - \bar{f})\, d\mu_\varepsilon + (\nu_\varepsilon - \mu_\varepsilon)(K)\bar{f}.$$

Using (7.44) and the fact that by construction $n\mu_\varepsilon(K) = \pm 2\pi m_\varepsilon(K)$, the sign being that of $\mu(K)$, we deduce

$$\left|\int_K f\, d\,(\nu_\varepsilon - \mu_\varepsilon)\right| \leq 2\pi\, \mathrm{osc}(f, K)\frac{m_\varepsilon(K)}{n} + \frac{2\pi}{n}\|f\|_\infty,$$

where $\mathrm{osc}(f, K) = \sup_{x,y\in K}|f(x) - f(y)|$. Summing over $K \in \mathcal{K}_\varepsilon$, using (7.43) and the fact that the number of squares is smaller than C/δ^2, we find

$$\left|\int f\, d(\nu_\varepsilon - \mu_\varepsilon)\right| \leq \frac{C}{n\delta^2}\|f\|_\infty + 2\pi\gamma_\varepsilon,$$

where $\gamma_\varepsilon = \sup_{K\in\mathcal{K}(\varepsilon)} \mathrm{osc}(f, K)$. But γ_ε is $o(1)$ as $\varepsilon \to 0$ since $\delta \ll 1$ and from (7.41) we have $1 \ll n\delta^2$. Therefore the right-hand side is $o(1)$ as $\varepsilon \to 0$, proving (7.47) and the claim. □

Proof of (7.39)–(7.40). It suffices to prove (7.39). Indeed, from (7.46) and the continuity of S_Ω in Ω we have

$$\lim_{\varepsilon\to 0}\frac{1}{n^2}\sum_{i,j}\iint S_\Omega(x,y)\,d\mu_i^\varepsilon(x)\,d\mu_j^\varepsilon(y)=\frac{4\pi^2}{\|\mu\|^2}\iint S_\Omega(x,y)\,d\mu(x)\,d\mu(y),$$

and the sum of the diagonal terms $i=j$ in the above double sum goes to zero, since it is less than C/n. Therefore

$$\lim_{\varepsilon\to 0}\frac{1}{n^2}\sum_{i\neq j}\iint S_\Omega(x,y)\,d\mu_i^\varepsilon(x)\,d\mu_j^\varepsilon(y)=\frac{4\pi^2}{\|\mu\|^2}\iint S_\Omega(x,y)\,d\mu(x)\,d\mu(y).$$

Adding to (7.39) yields (7.40), since $2\pi G_\Omega = S_\Omega - \log$.

We now prove (7.39). Given $\alpha>0$ and letting

$$\Delta_\alpha=\{(x,y)\mid |x-y|\le\alpha\},$$

by continuity of $\log|x-y|$ in $(\Omega\times\Omega)\setminus\Delta_\alpha$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{n^2}\sum_{i\neq j}\iint_{(\Delta_\alpha)^c}\log|x-y|\,d\mu_i^\varepsilon(x)\,d\mu_j^\varepsilon(y)$$
$$=\frac{4\pi^2}{\|\mu\|^2}\iint_{(\Delta_\alpha)^c}\log|x-y|\,d\mu(x)\,d\mu(y).$$

Therefore (7.39) will be proved if we show that

$$\lim_{\alpha\to 0}\limsup_{\varepsilon\to 0}\frac{1}{n^2}\sum_{i\neq j}\iint_{\Delta_\alpha}\log|x-y|\,d\mu_i^\varepsilon(x)\,d\mu_j^\varepsilon(y)=0, \tag{7.48}$$
$$\lim_{\alpha\to 0}\iint_{\Delta_\alpha}\log|x-y|\,d\mu(x)\,d\mu(y)=0.$$

The second limit is clearly equal to zero since $|\mu|\le C\mathbf{1}_\Omega\,dx$ and since $\log|x-y|$ is in $L^1(\Omega\times\Omega)$. The first limit follows from (7.45).

Indeed from (7.45) we may choose $\lambda>0$ such that if $i\neq j$, then

$$|a_i^\varepsilon-a_j^\varepsilon|\ge\frac{4\lambda}{\sqrt{n}}.$$

We may now define *disjoint* balls

$$B_i^\varepsilon = B\left(a_i^\varepsilon, \frac{\lambda}{\sqrt{n}}\right).$$

Moreover, if x and y belong respectively to B_i^ε and B_j^ε, then

$$\frac{|x-y|}{C} < |a_i^\varepsilon - a_j^\varepsilon| < C|x-y|, \tag{7.49}$$

and we already know that if x' and y' belong respectively to the support of μ_i^ε and the support of μ_j^ε,

$$\frac{|x'-y'|}{C} < |a_i^\varepsilon - a_j^\varepsilon| < C|x'-y'|. \tag{7.50}$$

This last inequality follows from the fact that μ_i^ε is supported in $\overline{B}(a_i^\varepsilon, c\varepsilon)$ for any i and $8c\varepsilon < |a_i^\varepsilon - a_j^\varepsilon|$, if $i \neq j$.

Using (7.49) and (7.50) we deduce that $|x-y| \leq C|x'-y'|$ and $|x'-y'| \leq C|x-y|$ for any $(x,y) \in B_i^\varepsilon \times B_j^\varepsilon$ and $(x',y') \in \overline{B}(a_i^\varepsilon, c\varepsilon) \times \overline{B}(a_j^\varepsilon, c\varepsilon)$. It follows that

$$\frac{1}{4\pi^2}\iint \big|\log|x'-y'|\big|\, d\mu_i^\varepsilon(x')\, d\mu_j^\varepsilon(y') \\ \leq \frac{1}{|B_i^\varepsilon \times B_j^\varepsilon|}\iint_{B_i^\varepsilon \times B_j^\varepsilon} |(\log|x-y|| + C)\, dx\, dy.$$

Summing over indices (i,j) such that the support of $\mu_i^\varepsilon \times \mu_j^\varepsilon$ intersects Δ_α and using the fact that $|B_i^\varepsilon \times B_j^\varepsilon| = C/n^2$ we deduce

$$\frac{1}{n^2}\sum_{i\neq j}\iint_{\Delta_\alpha} \big|\log|x'-y'|\big|\, d\mu_i^\varepsilon(x')\, d\mu_j^\varepsilon(y') \leq C \iint_{\Delta_{\alpha_\varepsilon}\cap\Omega} |(\log|x-y||+1)\, dx\, dy,$$

where $\alpha_\varepsilon = \alpha + C/\sqrt{n}$ converges to α as $\varepsilon \to 0$. Taking limsups with respect to ε and passing to the limit $\alpha \to 0$ yields (7.48), and then (7.39)–(7.40). $\square$

As an application, we obtain:

Proposition 7.5. *Assume $h_{ex}/|\log\varepsilon| \to \lambda > 0$ as $\varepsilon \to 0$. Given $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, there exists configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that $\|u_\varepsilon\|_{L^\infty} \le 1$ and*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \rightharpoonup \mu \quad \text{in } (C_0^{0,\gamma}(\Omega))^* \quad \forall \gamma \in (0,1) \tag{7.51}$$

and

$$\limsup_{\varepsilon\to 0} \frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{ex}}^2} \le \frac{\|\mu\|}{2\lambda} + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \tag{7.52}$$

where h_μ is given by (7.7).

Proof. We first assume $\mu \neq 0$ is a continuous and compactly supported function.

Let $n = \left[\frac{h_{\mathrm{ex}}}{2\pi}\|\mu\|\right]$, where $[\,\cdot\,]$ denotes the integer part, and apply Proposition 7.3 combined with Proposition 7.4. It yields the existence of $(u_\varepsilon, A_\varepsilon)$ such that $\frac{1}{n}\sum_{i=1}^n \mu_i^\varepsilon \rightharpoonup 2\pi\mu/\|\mu\|$ in the sense of measures, and therefore

$$\mu_\varepsilon := \frac{1}{h_{\mathrm{ex}}}\sum_{i=1}^n \mu_i^\varepsilon \rightharpoonup \mu,$$

with (7.23), (7.24), (7.40) and (7.25) holding. From (7.23) combined with (7.40), we have

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le \pi n|\log\varepsilon| + \frac{2\pi^2 n^2}{\|\mu\|^2}\iint G_\Omega(x,y)\,d\mu(x)\,d\mu(y) + o(n^2)$$

that is, inserting the particular choice of n,

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le \frac{h_{\mathrm{ex}}}{2}\|\mu\||\log\varepsilon| + \frac{{h_{\mathrm{ex}}}^2}{2}\iint G_\Omega(x,y)\,d\mu(x)\,d\mu(y) + o({h_{\mathrm{ex}}}^2).$$

Since $h_{\mathrm{ex}} = O(|\log\varepsilon|)$, we have $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le O(|\log\varepsilon|^2)$ and thus the remainder terms in (7.24) are $o(1)$, leading to

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le {h_{\mathrm{ex}}}^2 J_0 + \frac{h_{\mathrm{ex}}}{2}\|\mu\||\log\varepsilon| + {h_{\mathrm{ex}}}^2\int \xi_0\,d\mu_\varepsilon + \frac{{h_{\mathrm{ex}}}^2}{2}\iint G_\Omega(x,y)\,d\mu(x)\,d\mu(y) + o({h_{\mathrm{ex}}}^2). \tag{7.53}$$

Using the continuity of ξ_0 together with the convergence of μ_ε to μ, we deduce from (7.53) that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le {h_{\mathrm{ex}}}^2 J_0 + \frac{h_{\mathrm{ex}}}{2}\|\mu\|\,|\log\varepsilon| + {h_{\mathrm{ex}}}^2\int \xi_0\,d\mu \\ + \frac{{h_{\mathrm{ex}}}^2}{2}\iint G_\Omega(x,y)\,d\mu(x)\,d\mu(y) + o({h_{\mathrm{ex}}}^2). \quad (7.54)$$

But h_μ, U_μ and h_0 being defined respectively by (7.7), (7.19), and (7.1), we have $h_\mu = U_\mu + h_0$ and therefore $h_\mu - 1 = U_\mu + \xi_0$. Then expanding $\|U_\mu + \xi_0\|^2_{H^1(\Omega)}$ and using (7.20) we deduce

$$\frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2 = J_0 + \int \xi_0\,d\mu + \frac{1}{2}\iint G_\Omega(x,y)\,d\mu(x)\,d\mu(y).$$

Replacing in (7.54) and using $h_{\mathrm{ex}} \sim \lambda|\log\varepsilon|$ yields (7.52).

Moreover, since (7.25) holds and $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le O(|\log\varepsilon|^2)$, we have

$$\frac{1}{h_{\mathrm{ex}}}\,\|\mu(u_\varepsilon, A_\varepsilon) - \mu_\varepsilon\|_{(C_0^{0,\gamma}(\Omega))^*} \le C\varepsilon^\gamma\left(1 + \sqrt{\frac{F_\varepsilon(u_\varepsilon, A'_\varepsilon)}{n}}\right) \le o(1).$$

We conclude that (7.51) holds, which finishes the proof in the case where μ is a continuous and compactly supported function.

For the general case we use an approximation argument. Assume $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$ and consider a sequence $\{\mu_k\}_{k\in\mathbb{N}}$ of continuous functions compactly supported in Ω which converge to μ in the narrow sense of convergence of measures and in $H^{-1}(\Omega)$. In particular

$$\lim_{k\to+\infty} \frac{\|\mu_k\|}{2\lambda} + \frac{1}{2}\int_\Omega |\nabla h_{\mu_k}|^2 + |h_{\mu_k} - 1|^2 \\ = \frac{\|\mu\|}{2\lambda} + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2. \quad (7.55)$$

We may then apply the proposition to each μ_k, and get configurations $\{u_\varepsilon^k, A_\varepsilon^k)\}_\varepsilon$ which satisfy (7.51)–(7.52), with μ_k instead of μ.

Then, a diagonal argument, together with (7.55), yields a sequence $\varepsilon_k \to 0$—that we write in shorthand $\{\varepsilon\}$—such that, writing $(u_\varepsilon, A_\varepsilon)$ instead of $(u_{\varepsilon_k}^k, A_{\varepsilon_k}^k)$, both (7.51) and (7.52) hold. □

Remark 7.2. *In Chapter 8 we will obtain a sharper upper bound in the case* $\lambda = +\infty$, *i.e.,* $h_{ex} \gg |\log \varepsilon|$. *Observe also that an upper bound* $\min G_\varepsilon \le \frac{1}{4\varepsilon^2}$ *(useful when* $h_{ex} \ge \frac{C}{\varepsilon^2}$*) is easy to obtain by considering the normal configuration* $u \equiv 0, h \equiv h_{ex}$ *(see Chapter 2).*

7.4 Proof of Theorems 7.1 and 7.2

Item 2) of Theorem 7.1, stated above as Proposition 7.5, has already been proved.

7.4.1 Proof of Theorem 7.1, Item 1)

We assume $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C{h_{\rm ex}}^2$ and $\|u_\varepsilon\|_\infty \le 1$. Then from Lemma 3.4,

$$|\nabla_{A_\varepsilon} u_\varepsilon|^2 \ge |u_\varepsilon|^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 \ge |j_\varepsilon|^2,$$

with $j_\varepsilon = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)$. From the upper bound $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C{h_{\rm ex}}^2$, we deduce that $j_\varepsilon / h_{\rm ex}$ and $h_\varepsilon / h_{\rm ex}$ are bounded in $L^2(\Omega)$, hence up to extraction they converge weakly in $L^2(\Omega)$ as $\varepsilon \to 0$ to some j, h. Moreover, since $\mu(u_\varepsilon, A_\varepsilon) = \operatorname{curl} j_\varepsilon + h_\varepsilon$, we have

$$\mu_\varepsilon := \frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}} \to \mu = \operatorname{curl} j + h$$

weakly in $H^{-1}(\Omega)$. It remains to prove the convergence of μ_ε in $(C_0^{0,\gamma}(\Omega))^*$ and the lower bound (7.8).

Since $h_{\rm ex} \le C|\log \varepsilon|$, we have $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C{h_{\rm ex}}^2 \le C|\log \varepsilon|^2$. We deduce that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log \varepsilon|^2$ too since $F_\varepsilon(u, A) \le 2G_\varepsilon(u, A) + 2{h_{\rm ex}}^2|\Omega|$, which follows from $(h - h_{\rm ex})^2 \le 2(h^2 + {h_{\rm ex}}^2)$.

Let U be an open subdomain of Ω. Working in U rather than in Ω will be useful in order to prove (7.9) and (7.10) in Theorem 7.2 in the next section. Applying Theorem 4.1 in U with $r \ll 1$ such that $|\log r| \ll |\log \varepsilon|$, we get a family of disjoint closed balls $B(a_i, r_i)$ with $\sum_i r_i \le Cr$. We call the union of these balls V_ε. From Theorem 4.1 we have

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, V_\varepsilon) \ge \pi \sum_i |d_i| \left(\log \frac{1}{\varepsilon \sum_i |d_i|} - o(|\log \varepsilon|) \right).$$

Moreover, since $F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log \varepsilon|^2$ and from (4.4) we have $\sum_i |d_i| \le C|\log \varepsilon|$, hence the above may be rewritten, since $|\log \varepsilon|$ and $h_{\rm ex}$ are comparable

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, V_\varepsilon) \geq \pi \sum_i |d_i||\log\varepsilon| - o({h_{\rm ex}}^2). \tag{7.56}$$

In the above we want to replace F_ε with G_ε. The difference between the two is the integral of ${h_{\rm ex}}^2 - 2hh_{\rm ex}$ over V_ε which, using Cauchy–Schwarz, is less than $|V_\varepsilon|{h_{\rm ex}}^2 + \sqrt{|V_\varepsilon|}h_{\rm ex}\|h_\varepsilon\|_{L^2}$, hence is $o({h_{\rm ex}}^2)$ since $\|h_\varepsilon\|_{L^2} \leq Ch_{\rm ex}$ and the area of V_ε tends to 0. Therefore, using (7.56)

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A_\varepsilon, U) &= G_\varepsilon(u_\varepsilon, A_\varepsilon, V_\varepsilon) + G_\varepsilon(u_\varepsilon, A_\varepsilon, U \setminus V_\varepsilon) \\ &\geq \pi \sum_i |d_i||\log\varepsilon| + G_\varepsilon(u_\varepsilon, A_\varepsilon, U \setminus V_\varepsilon) - o({h_{\rm ex}}^2), \end{aligned} \tag{7.57}$$

and then, dividing by ${h_{\rm ex}}^2$,

$$\begin{aligned} &\frac{G_\varepsilon(u_\varepsilon, A_\varepsilon, U)}{{h_{\rm ex}}^2} \\ &\qquad \geq \pi \frac{\sum_i |d_i|}{h_{\rm ex}} \frac{|\log\varepsilon|}{h_{\rm ex}} + \int_{U\setminus V_\varepsilon} \left|\frac{j_\varepsilon}{h_{\rm ex}}\right|^2 + \left|\frac{h_\varepsilon}{h_{\rm ex}} - 1\right|^2 - o(1). \end{aligned} \tag{7.58}$$

Let us now examine the limit as $\varepsilon \to 0$ of the right-hand side of this inequality. Since $\sum_i r_i \to 0$ we may extract a sequence $\varepsilon_n \to 0$ such that, denoting

$$\mathcal{A}_N := \cup_{n\geq N} V_{\varepsilon_n},$$

we have $|\mathcal{A}_N| \to 0$ as $N \to \infty$. By weak convergence of $j_{\varepsilon_n}/h_{\rm ex}$ and $h_{\varepsilon_n}/h_{\rm ex}$, we have for any fixed N

$$\begin{aligned} \liminf_{n\to\infty} \int_{U\setminus V_{\varepsilon_n}} \left|\frac{j_{\varepsilon_n}}{h_{\rm ex}}\right|^2 + \left|\frac{h_{\varepsilon_n}}{h_{\rm ex}} - 1\right|^2 &\geq \liminf_{n\to\infty} \int_{U\setminus \mathcal{A}_N} \left|\frac{j_{\varepsilon_n}}{h_{\rm ex}}\right|^2 + \left|\frac{h_{\varepsilon_n}}{h_{\rm ex}} - 1\right|^2 \\ &\geq \int_{U\setminus\mathcal{A}_N} |j|^2 + |h-1|^2. \end{aligned}$$

Passing to the limit $N \to \infty$, we find

$$\liminf_{n\to\infty} \int_{U\setminus V_{\varepsilon_n}} \left|\frac{j_{\varepsilon_n}}{h_{\rm ex}}\right|^2 + \left|\frac{h_{\varepsilon_n}}{h_{\rm ex}} - 1\right|^2 \geq \int_U |j|^2 + |h-1|^2. \tag{7.59}$$

On the other hand, returning to (7.57) and using the a priori bound $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C{h_{\rm ex}}^2$, we find that $(1/h_{\rm ex})\sum_i |d_i|$ remains bounded. Thus, $(2\pi/h_{\rm ex})\sum_i d_i\delta_{a_i}$ is weakly compact in the sense of measures, and we may assume it converges to a measure in $(C^0_0(U))^*$. Using the Jacobian estimate Theorem 6.1 in U, this limit is also the limit of μ_ε, i.e., is $\mu = \operatorname{curl} j + h$ or to be precise the restriction of μ to U. Note that Theorem 6.2 applied in Ω implies that μ_ε converges to μ in $(C^{0,\gamma}_0(\Omega))^*$. Passing to the limit in (7.58) and inserting (7.59), we find

$$\liminf_{n\to\infty} \frac{G_\varepsilon(u_{\varepsilon_n}, A_{\varepsilon_n}, U)}{{h_{\rm ex}}^2} \ge \frac{1}{2\lambda}|\mu|(U) + \frac{1}{2}\int_U |j|^2 + |h-1|^2. \tag{7.60}$$

Denoting by h_μ the solution of (7.7), writing j as $-\nabla^\perp h_\mu + (j+\nabla^\perp h_\mu)$ and h as $h_\mu + (h-h_\mu)$, and observing that

$$\operatorname{curl}(j+\nabla^\perp h_\mu) + h - h_\mu = 0, \tag{7.61}$$

we have

$$\begin{aligned}\int_\Omega |j|^2 + |h-1|^2 &= \int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2 + |j+\nabla^\perp h_\mu|^2 + |h-h_\mu|^2 \\ &\quad + 2\int_\Omega (-\nabla^\perp h_\mu)\cdot(j+\nabla^\perp h_\mu) + (h_\mu - 1)(h-h_\mu) \\ &= \int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2 + |j+\nabla^\perp h_\mu|^2 + |h-h_\mu|^2\end{aligned}$$

where we have used an integration by parts and (7.61). Inserting this into (7.60) with $U=\Omega$, we deduce that (7.8) holds, completing the proof of Theorem 7.1, item 1).

7.4.2 Proof of Theorem 7.2

Combining 1) and 2) of Theorem 7.1, if $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ are minimizers of G_ε we must have

$$\lim_{\varepsilon\to 0} \frac{\min G_\varepsilon}{{h_{\rm ex}}^2} = \min E_\lambda, \tag{7.62}$$

together with $j = -\nabla^\perp h_\mu$, and $h = h_\mu$. Hence the vorticity of minimizers of G_ε must converge, after extraction, to the unique minimizer μ_* of E_λ.

The uniqueness of μ_* implies that the whole sequence $\mu(u_\varepsilon, A_\varepsilon)$ converges to μ_* and $h_\varepsilon/h_{\rm ex}$ to h_{μ_*}. Also, since $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε, it is a critical point and solves the second Ginzburg–Landau equation

$$-\nabla^\perp h_\varepsilon = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) = j_\varepsilon, \tag{7.63}$$

which implies that $|\nabla h_\varepsilon| \le |\nabla_{A_\varepsilon} u_\varepsilon|$ (see Lemma 3.3) and thus $h_\varepsilon/h_{\rm ex}$ is bounded in $H^1(\Omega)$. Taking the curl of (7.63) we also deduce that $-\Delta h_\varepsilon + h_\varepsilon = \mu(u_\varepsilon, A_\varepsilon)$. Since $(C^{0,\gamma}(\Omega))^*$ convergence is stronger than $W^{-1,p}(\Omega)$ convergence for $p < 2$, by elliptic regularity we deduce that $h_\varepsilon/h_{\rm ex}$ converges strongly in $W^{1,p}(\Omega)$ for $p < 2$, and weakly in $H^1(\Omega)$.

Returning to (7.60), we have for any open subdomain U of Ω

$$\liminf_{n\to\infty} \frac{G_\varepsilon(u_{\varepsilon_n}, A_{\varepsilon_n}, U)}{{h_{\rm ex}}^2} \ \ge\ \frac{1}{2\lambda}|\mu_*|(U) + \frac{1}{2}\int_U |\nabla h_{\mu_*}|^2 + |h_{\mu_*} - 1|^2,$$

but there must be equality from (7.62), therefore since this is true for any U, (7.9) holds.

From (7.9), the fact that $|\nabla_{A_\varepsilon} u_\varepsilon| \ge |j_\varepsilon| = |\nabla h_\varepsilon|$ and the strong L^2 convergence of h_ε, we have

$$\liminf_{\varepsilon\to 0} \frac{|\nabla_{A_\varepsilon} u_\varepsilon|^2}{{h_{\rm ex}}^2} \ge \liminf_{\varepsilon\to 0} \frac{|\nabla h_\varepsilon|^2}{{h_{\rm ex}}^2} \ge |\nabla h_{\mu_*}|^2 + \frac{1}{\lambda}\mu_*$$

and conversely, from the energy upper-bound,

$$\frac{1}{{h_{\rm ex}}^2}\int_\Omega |\nabla h_\varepsilon|^2 \le \frac{\|\mu_*\|}{\lambda} + \int_\Omega |\nabla h_{\mu_*}|^2.$$

Combining the two, we must have

$$\frac{|\nabla h_\varepsilon|^2}{{h_{\rm ex}}^2} \to |\nabla h_{\mu_*}|^2 + \frac{1}{\lambda}\mu_*$$

as measures, and (7.10) follows from the Brezis–Lieb lemma.

Bibliographic notes on Chapter 7: The results of this chapter were for the most part obtained in [168]: the statement there, was not exactly in the Γ-convergence framework, however the structure of the proof essentially was. A version with pinning term in the energy can be found in [5] (some of the presentation here is borrowed more from [5]). Analogous results for the Ginzburg–Landau energy without magnetic field, dealing also with nonsimply connected domains, were given by Jerrard–Soner in [118].

Concerning the derivation of the value of $H^0_{c_1}$, the result in (7.16) confirms the formal derivation in the physics literature (Abrikosov [1], DeGennes [80]), and by Bethuel–Rivière in [51].

Chapter 8

Higher Values of the Applied Field

The previous chapter dealt with minimizers of the Ginzburg–Landau functional when the applied field was $O(|\log \varepsilon|)$. The applied field behaving asymptotically like $\lambda|\log \varepsilon|$, letting $\lambda \to \infty$ in Theorem 7.2 indicates that for energy-minimizers for applied fields $h_{\text{ex}} \gg |\log \varepsilon|$, we must have $\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\text{ex}}} \to 1$, and $\frac{h_\varepsilon}{h_{\text{ex}}} \to 1$. But in this regime, $\frac{G_\varepsilon(u_\varepsilon, A_\varepsilon)}{{h_{\text{ex}}}^2} \to 0$ and the arguments of Chapter 7 do not give, even formally, the leading order term of the minimal energy. Moreover, the tools which were at the heart of the result, namely the vortex balls construction of Theorem 4.1 and the Jacobian estimate of Theorem 6.1 break down for higher values of h_{ex}.

On the other hand, we recall from Chapter 2 the prediction by Abrikosov that the transition from the mixed state, which we may as well call the vortex state, to the normal state, should occur for $h_{\text{ex}} \approx 1/\varepsilon^2$, i.e., much higher fields. We will show in this chapter how our techniques still allow us to find the minimum of the energy for applied fields satisfying $|\log \varepsilon| \ll h_{\text{ex}} \ll 1/\varepsilon^2$: in the scaling of Chapter 7 what we determine here is the first nonzero lower-order correction term. We find that minimizers have a uniform limiting density in the whole domain Ω, in agreement with Abrikosov lattices. In fact, the test-configurations we use below to obtain the upper bound on the minimal energy are constructed to be periodic.

Theorem 8.1. *Assume, as $\varepsilon \to 0$, that $|\log \varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$. Then, letting $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, and letting $g_\varepsilon(u, A)$ denote the energy-density*

$\frac{1}{2}\left(|\nabla_A u|^2 + |h-h_{ex}|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$, *we have*

$$\frac{2g_\varepsilon(u_\varepsilon, A_\varepsilon)}{h_{ex}\log\frac{1}{\varepsilon\sqrt{h_{ex}}}} \rightharpoonup dx \quad \text{as } \varepsilon \to 0 \tag{8.1}$$

in the weak sense of measures, where dx denotes the 2-dimensional Lebesgue measure, and

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u,A) \sim \frac{|\Omega|}{2} h_{ex}\log\frac{1}{\varepsilon\sqrt{h_{ex}}} \quad \text{as } \varepsilon\to 0, \tag{8.2}$$

where $|\Omega|$ is the area of Ω.

Since in this regime $h_{\text{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}} \ll {h_{\text{ex}}}^2$, we deduce as an immediate corollary:

Corollary 8.1. *Assume that, as $\varepsilon \to 0$, $|\log\varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ and $(u_\varepsilon, A_\varepsilon)$ minimize G_ε, letting $h_\varepsilon = \operatorname{curl} A_\varepsilon$ and $\mu(u_\varepsilon, A_\varepsilon) = \operatorname{curl}(iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + h_\varepsilon$, we have*

$$\frac{h_\varepsilon}{h_{ex}} \to 1 \quad \text{in } H^1(\Omega)$$
$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{ex}} \to dx \quad \text{in } H^{-1}(\Omega).$$

Proof. Since $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε, it is a solution of (GL) and thus, using Lemma 3.3, we find

$$\|h_\varepsilon - h_{\text{ex}}\|^2_{H^1(\Omega)} \le 2G_\varepsilon(u_\varepsilon, A_\varepsilon) \ll {h_{\text{ex}}}^2$$

hence $h_\varepsilon/h_{\text{ex}} \to 1$ in $H^1(\Omega)$. Since we have the relation $-\Delta h_\varepsilon + h_\varepsilon = \mu(u_\varepsilon, A_\varepsilon)$ obtained by taking the curl of the second Ginzburg–Landau equation, the convergence of $\mu(u_\varepsilon, A_\varepsilon)/h_{\text{ex}}$ follows. □

The theorem is a direct consequence of Propositions 8.1 and 8.2 below, but let us briefly explain what problem occurs for high fields and how it is overcome. If h_{ex} is too high, say $h_{\text{ex}} \gg 1/\varepsilon$, then a minimizer of G_ε is expected to have a number of vortices n of the order of h_{ex} and then the perimeter of the set where $|u| < 1/2$ should be of the order $n\varepsilon \gg 1$. This means that we can no longer hope that the a priori bound on the energy satisfied by a minimizer excludes, say, a line where $|u| = 0$. As we mentioned, the downside is that the vortex balls construction and the Jacobian estimate, which are based on covering the set $\{|u| = 0\}$ by small balls, will not work anymore.

On the other hand, for such large fields, the problem of minimizing G_ε reduces to that of minimizing it on any subdomain, in other words the minimization problem becomes *local*. Thus we may perform blow-ups which yield the right lower bound. The effect of the blow-ups will be precisely to effectively reduce $h_{\rm ex}$ and allow our techniques to be applied on the smaller scale. On the other hand, that the upper bound that we need will demand a more rigid construction of a good test-configuration than in Proposition 7.4.

The rescaling formula is:

Lemma 8.1. *Given (u, A) and Ω, assuming $0 \in \Omega$, define u_λ, A_λ and Ω_λ by*

$$u_\lambda(\lambda x) = u(x), \quad \lambda A_\lambda(\lambda x) = A(x), \quad \Omega_\lambda = \lambda\Omega. \tag{8.3}$$

Then, for any h_{ex}, we have $G_\varepsilon(u, A, \Omega) = G_\varepsilon^\lambda(u_\lambda, A_\lambda, \Omega_\lambda)$, where

$$G_\varepsilon^\lambda(u_\lambda, A_\lambda, \Omega_\lambda) = \frac{1}{2}\int_{\Omega_\lambda} |\nabla_{A_\lambda} u_\lambda|^2 + \lambda^2 \left(\operatorname{curl} A_\lambda - \frac{h_{ex}}{\lambda^2}\right)^2 + \frac{\left(1 - |u_\lambda|^2\right)^2}{2(\lambda\varepsilon)^2}. \tag{8.4}$$

The proof is straightforward and we omit it.

8.1 Upper Bound

Proposition 8.1. *Assume, as $\varepsilon \to 0$, that $1 \ll h_{ex} \ll 1/\varepsilon^2$. Then for any ε small enough*

$$\min_{(u,A)\in H^1\times H^1} G_\varepsilon(u, A, \Omega) \le h_{ex}\frac{|\Omega|}{2}\left(\log\frac{1}{\varepsilon\sqrt{h_{ex}}} + C\right). \tag{8.5}$$

Proof. The proof is done by constructing a test configuration $(u_\varepsilon, A_\varepsilon)$ which is periodic, in the sense that gauge-invariant quantities are periodic. Let

$$\lambda = \sqrt{\frac{h_{\rm ex}}{2\pi}}$$

and let $L_\varepsilon = \lambda\mathbb{Z} \times \lambda\mathbb{Z}$. We let h_ε be the solution in $\mathbb{R}^2$ of

$$-\Delta h_\varepsilon + h_\varepsilon = 2\pi \sum_{a \in L_\varepsilon} \delta_a. \tag{8.6}$$

It is thus periodic with respect to L_ε.

Then we define ρ_ε by

$$\rho_\varepsilon(x) = \begin{cases} 0 & \text{if } |x-a| \le \varepsilon \text{ for some } a \in L_\varepsilon, \\ \dfrac{|x-a|}{\varepsilon} - 1 & \text{if } \varepsilon < |x-a| < 2\varepsilon \text{ for some } a \in L_\varepsilon, \\ 1 & \text{otherwise.} \end{cases} \tag{8.7}$$

Finally, as in the proof of Proposition 7.3, we define A_ε to solve $\operatorname{curl} A_\varepsilon = h_\varepsilon$ and φ_ε, well defined modulo 2π, to solve $-\nabla^\perp h_\varepsilon = \nabla\varphi_\varepsilon - A_\varepsilon$ in $\mathbb{R}^2 \setminus L_\varepsilon$. Then we let $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$.

By construction, every gauge-invariant quantity is periodic with respect to the lattice L_ε, thus if we choose the origin carefully, the energy $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ will be estimated by computing the energy per unit cell. Indeed, let

$$K_\varepsilon = \left(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right) \times \left(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right)$$

be the unit cell of L_ε. For each $x \in K_\varepsilon$ we may define a translated lattice L_ε^x, and a corresponding test configuration $(u_\varepsilon^x, A_\varepsilon^x)$, with energy density $\mathrm{gl}_\varepsilon^x(y) = \mathrm{gl}_\varepsilon(y-x)$. Then, applying Fubini's theorem, we have

$$\int_{x \in K_\varepsilon} G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) dx = \iint_{\substack{x \in K_\varepsilon \\ y \in \Omega}} \mathrm{gl}_\varepsilon^x(y)\, dx\, dy = |\Omega| G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon),$$

since gl_ε is periodic with respect to the lattice L_ε. It follows, using the mean value formula, that we may choose x such that

$$G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) \le \frac{|\Omega|}{|K_\varepsilon|} G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon). \tag{8.8}$$

We estimate $G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon)$, arguing as in Proposition 7.3: we have $|\nabla_{A_\varepsilon} u_\varepsilon|^2 = |\nabla\rho_\varepsilon|^2 + {\rho_\varepsilon}^2 |\nabla\varphi_\varepsilon - A_\varepsilon|^2$ and ${\rho_\varepsilon}^2 |\nabla\varphi_\varepsilon - A_\varepsilon|^2 \ge |\nabla h_\varepsilon|^2$. Moreover, writing B_r for $B(0,r)$ and using (8.7)

$$\frac{1}{2} \int_{B_{2\varepsilon}} |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2} \left(1 - {\rho_\varepsilon}^2\right)^2 \le C.$$

We deduce that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, K_\varepsilon) \le \frac{1}{2} \int_{K_\varepsilon \setminus B_\varepsilon} |\nabla h_\varepsilon|^2 + \frac{1}{2} \int_{K_\varepsilon} (h_\varepsilon - h_{\text{ex}})^2 \, dx + C. \tag{8.9}$$

To estimate the right-hand side, we perform a change of variables $y = \lambda x$. Then

$$\int_{K_\varepsilon \setminus B_\varepsilon} |\nabla h_\varepsilon|^2 + \int_{K_\varepsilon} (h_\varepsilon - h_{\text{ex}})^2 \, dx = \int_{K \setminus B_{\lambda\varepsilon}} |\nabla \tilde{h}_\varepsilon|^2 + \frac{2\pi}{h_{\text{ex}}} \int_K \tilde{h}_\varepsilon^2 \, dy \tag{8.10}$$

where $\tilde{h}_\varepsilon(y) = h_\varepsilon(x) - h_{\text{ex}}$ and $K = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$. Now we decompose $\tilde{h}_\varepsilon - h_{\text{ex}}$ as

$$\tilde{h}_\varepsilon(y) = g_\varepsilon(y) - \log|y|, \tag{8.11}$$

and we show that g_ε is bounded in $W^{1,q}(K)$ independently of ε for any $q > 0$.

First, by periodicity, the integral of h_ε in K_ε is 2π, thus the integral of $\tilde{h}_\varepsilon$ in K is $2\pi\lambda^2 - h_{\text{ex}} = 0$. Therefore g_ε and $\log|\cdot|$ have the same mean value in K, and that value does not depend on ε. We deduce from Poincaré's inequality that

$$\|g_\varepsilon\|_{L^2(K)}^2 \le C \left(1 + \|\nabla g_\varepsilon\|_{L^2(K)}^2\right). \tag{8.12}$$

Second note that h_ε, which is the solution to (8.6), is also the solution of $-\Delta h_\varepsilon + h_\varepsilon = 2\pi\delta_0$ in K_ε and $\partial_\nu h_\varepsilon = 0$ on ∂K_ε. Indeed, the problem (8.6) is symmetric with respect to each line containing a side of the square K_ε, hence h_ε is equal to its symmetrized and $\partial_\nu h_\varepsilon = 0$ on ∂K_ε. Therefore $g_\varepsilon(y) = h_\varepsilon(y/\lambda) - h_{\text{ex}} + \log|y|$ solves

$$\begin{cases} -\Delta g_\varepsilon + \lambda^{-2}\left(g_\varepsilon + h_{\text{ex}} - \log\right) = 0 & \text{in } K, \\ \partial_\nu g_\varepsilon = \partial_\nu \log & \text{on } \partial K. \end{cases}$$

Multiplying the equation by g_ε and integrating by parts in K yields

$$\int_K |\nabla g_\varepsilon|^2 + \frac{1}{\lambda^2}\left(g_\varepsilon^2 + g_\varepsilon h_{\text{ex}} - g_\varepsilon \log\right) = \int_{\partial K} g_\varepsilon \partial_\nu g_\varepsilon.$$

We deduce, replacing λ by its value and using the facts that $\partial_\nu g_\varepsilon = \partial_\nu \log$ on ∂K and that the average of g_ε on K does not depend on ε,

$$\|\nabla g_\varepsilon\|_{L^2(K)}^2 \leq C\left(1 + {h_{\mathrm{ex}}}^{-1}\|g_\varepsilon\|_{L^2(K)}^2 + \|g_\varepsilon\|_{L^2(\partial K)}\right). \tag{8.13}$$

Since $1 \ll h_{\mathrm{ex}}$, if ε is small enough, then h_{ex} is large enough so that using (8.12) and bounding the L^2 norm of the trace of g_ε by the H^1 norm, the terms in the right-hand side of (8.13) are absorbed by $\|\nabla g_\varepsilon\|_{L^2(K)}^2$ yielding $\|g_\varepsilon\|_{H^1(K)} \leq C$. We deduce that g_ε is bounded independently of ε in $L^q(K)$ for every $q > 0$ and then, using the equation satisfied by g_ε, that for every $q > 0$

$$\|\nabla g_\varepsilon\|_{W^{1,q}(K)}^2 \leq C.$$

Together with (8.11), this implies that

$$\int\limits_{K\setminus B_{\lambda\varepsilon}} |\nabla \tilde{h}_\varepsilon|^2 \leq C + \int\limits_{K\setminus B_{\lambda\varepsilon}} |\nabla \log|^2 \leq C + 2\pi \log\frac{1}{\lambda\varepsilon},$$

and also $\frac{2\pi}{h_{\mathrm{ex}}}\int_K \tilde{h}_\varepsilon^2 \leq C$. Together with (8.8), (8.9), and (8.10), this yields, since $|K_\varepsilon| = \lambda^{-2} = 2\pi/h_{\mathrm{ex}}$,

$$G_\varepsilon\left(u_\varepsilon^x, A_\varepsilon^x, \Omega\right) \leq \frac{|\Omega|}{|K_\varepsilon|}\left(\pi\log\frac{1}{\lambda\varepsilon} + C\right) \leq \frac{|\Omega|}{2}h_{\mathrm{ex}}\left(\log\frac{1}{\sqrt{h_{\mathrm{ex}}}\varepsilon} + C\right). \qquad \square$$

8.2 Lower Bound

We now wish to compute a lower bound for $G_\varepsilon(u, A)$ which matches the upper bound of the previous section. In the course of the proof we will see clearly that if (u, A) minimizes G_ε, then its energy is accounted for by the vortex-energy.

In what follows we denote $B_\lambda^x = B(x, \lambda^{-1})$ and we will often omit the subscript ε, where x is the center of the blow-up.

Proposition 8.2. *Assume $|\log\varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ and $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε. Then there exists $1 \ll \lambda \ll \frac{1}{\varepsilon}$ such that for every $x \in \Omega$ such that $B_\lambda^x \subset \Omega$, we have*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, B_\lambda^x) \geq \frac{|B_\lambda^x|}{2} h_{ex}\log\frac{1}{\varepsilon\sqrt{h_{ex}}}\left(1 - o(1)\right). \tag{8.14}$$

Proof. As already mentioned, the proof is achieved by blowing up at the scale λ.

Define u_λ and A_λ as in (8.3), but taking the origin at x. From Lemma 8.1, (8.4), again with the origin at x, and dropping the ε subscripts, the left-hand side of (8.14) is equal to

$$\frac{1}{2}\int_{B_1} |\nabla_{A_\lambda} u_\lambda|^2 + \lambda^2 \left(\operatorname{curl} A_\lambda - \frac{h_{\mathrm{ex}}}{\lambda^2}\right)^2 + \frac{\left(1-|u_\lambda|^2\right)^2}{2(\lambda\varepsilon)^2}$$

thus, letting $u' = u_\lambda$, $A' = A_\lambda$, $\varepsilon' = \lambda\varepsilon$ and ${h_{\mathrm{ex}}}' = h_{\mathrm{ex}}/\lambda^2$, the inequality (8.14) that we wish to prove is equivalent to

$$\frac{1}{2}\int_{B_1} |\nabla_{A'} u'|^2 + \lambda^2 \left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{|B_1|}{2} {h_{\mathrm{ex}}}' \log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}}\left(1-o(1)\right). \quad (8.15)$$

Now we choose λ such that

$${h_{\mathrm{ex}}}' = |\log\varepsilon'|. \quad (8.16)$$

Let us check that this is possible and give the behavior of λ as $\varepsilon \to 0$. Condition (8.16) is equivalent to $\varepsilon^2 h_{\mathrm{ex}} = f(\varepsilon\lambda)$, where $f(x) = x^2\log(1/x)$. Since $\varepsilon^2 h_{\mathrm{ex}} \to 0$ as $\varepsilon \to 0$, it is easy to check that for ε small enough, there is a unique $x_\varepsilon \in (0, 1/2)$ satisfying $f(x_\varepsilon) = \varepsilon^2 h_{\mathrm{ex}}$. Moreover from $|\log\varepsilon| \ll h_{\mathrm{ex}} \ll 1/\varepsilon^2$ we deduce $\varepsilon \ll x_\varepsilon \ll 1$. Therefore (8.16) can indeed be verified, and the corresponding λ, ε' satisfy

$$1 \ll \lambda \ll \frac{1}{\varepsilon}, \quad \varepsilon' \ll 1, \quad \log\frac{1}{\varepsilon\sqrt{h_{\mathrm{ex}}}} \approx |\log\varepsilon'|,$$

the last identity being deduced from $\varepsilon^2 h_{\mathrm{ex}} = f(\varepsilon\lambda) = f(\varepsilon')$ by taking logarithms. Thus with this choice of λ, (8.15) becomes

$$\frac{1}{2}\int_{B_1} |\nabla_{A'} u'|^2 + \lambda^2 \left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2\varepsilon'^2} \geq \frac{|B_1|}{2} {h_{\mathrm{ex}}}' |\log\varepsilon'| \left(1-o(1)\right). \quad (8.17)$$

Two cases may now occur, depending on the blow-up origin x. Either

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \gg {h_{\mathrm{ex}}}'^2$$

as $\varepsilon \to 0$ and then, from (8.16), (8.17) is clearly satisfied, or

$$\frac{1}{2}\int_{B_1} |\nabla_{A'}u'|^2 + \lambda^2\left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \le C{h_{\mathrm{ex}}}'^2.$$

This way, we have reduced to the case of configurations with a relatively small energy, for which all the analysis of previous chapters apply.

In this case, since $\lambda \gg 1$ we find

$$\frac{\operatorname{curl} A' - {h_{\mathrm{ex}}}'}{{h_{\mathrm{ex}}}'} \to 0, \quad \text{in } L^2(B_1). \tag{8.18}$$

On the other hand, replacing ε by ε' and h_{ex} by ${h_{\mathrm{ex}}}'$, the hypotheses of Theorem 7.1, item 1) are satisfied and we deduce from (7.6), (7.8) that

$$\liminf_{\varepsilon' \to 0} \frac{1}{2{h_{\mathrm{ex}}}'^2}\int_{B_1} |\nabla_{A'}u'|^2 + \left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \ge \frac{\|\mu'\|}{2},$$

where $\mu' = -\Delta h' + h'$ and h' is the limit of $\operatorname{curl} A'/{h_{\mathrm{ex}}}'$. From (8.18) we have $\mu' = 1$, hence

$$\liminf_{\varepsilon' \to 0} \frac{1}{2{h_{\mathrm{ex}}}'^2}\int_{B_1} |\nabla_{A'}u'|^2 + \left(\operatorname{curl} A' - {h_{\mathrm{ex}}}'\right)^2 + \frac{\left(1-|u'|^2\right)^2}{2{\varepsilon'}^2} \ge \frac{\pi}{2},$$

and (8.17) is satisfied since for our choice of λ

$$\frac{\pi}{2}{h_{\mathrm{ex}}}'^2 = \frac{|B_1|}{2}{h_{\mathrm{ex}}}'\log\frac{1}{\varepsilon'}.$$

We have shown for our particular choice of λ that (8.17), hence (8.15) and then (8.14) are satisfied for every choice of blow-up origin x. $\square$

To conclude the proof of Theorem 8.1, we integrate (8.14) with respect to x. Letting U be any open subdomain of Ω, using Fubini's theorem, we have

$$\int_{x\in U} G_\varepsilon(u,A,B_\lambda^x\cap U) = \iint_{\substack{x\in U\\ y\in B_\lambda^x\cap U}} g_\varepsilon(u,A)(y)\,dy\,dx$$
$$= \iint_{\substack{x\in U\\ y\in B_\lambda^x\cap U}} g_\varepsilon(u,A)(y)\,dx\,dy$$
$$= \int_{y\in U} |B_\lambda^y\cap U| g_\varepsilon(u,A)(y)\,dy \le \frac{\pi}{\lambda^2} G_\varepsilon(u,A,U).$$

We deduce that

$$\liminf_{\varepsilon\to 0} \frac{G_\varepsilon(u,A,U)}{h_{\text{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}} \ge \liminf_{\varepsilon\to 0}\int_{x\in U} \frac{\lambda^2 G_\varepsilon(u,A,B_\lambda^x\cap U)}{\pi h_{\text{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}}$$
$$\ge \liminf_{\varepsilon\to 0}\int_{x\in U, B_\lambda^x\subset U} \frac{\lambda^2 G_\varepsilon(u,A,B_\lambda^x\cap U)}{\pi h_{\text{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}}$$
$$\ge \int_{x\in U}\liminf_{\varepsilon\to 0}\left(\mathbf{1}_{B_\lambda^x\subset U}\frac{G_\varepsilon(u,A,B_\lambda^x)}{h_{\text{ex}}|B_\lambda^x|\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}}\right)$$
$$\ge \frac{|U|}{2}, \tag{8.19}$$

where we have used Fatou's lemma and (8.14). In view of Proposition 8.1, we know that $\left(h_{\text{ex}}\log\frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}\right)^{-1} g_\varepsilon(u_\varepsilon,A_\varepsilon)$ is bounded in $L^1(\Omega)$, hence has a weak limit g in the sense of measures. Since continuous functions on Ω can be uniformly approximated by characteristic functions, (8.19) allows to say that $g\ge\frac{dx}{2}$. But since (8.5) holds, there must be equality, which proves (8.1), and (8.2) immediately follows.

Bibliographic notes on Chapter 8: The result of this chapter was obtained in [170], but the proof is presented here under a much simpler form. The case of higher h_{ex}, of order b/ε^2 with $b<1$, was studied in [172].

Chapter 9

The Intermediate Regime

When $h_{\rm ex} \sim H^0_{c_1}$ i.e., $\frac{h_{\rm ex}}{|\log \varepsilon|} \to \lambda = \frac{1}{2|\underline{\xi_0}|}$, then from Theorem 7.2, we get that the limiting minimizer is h_0 hence $\mu_* = 0$. Moreover, comparing the lower bounds (7.58) and (7.59) to the upper bound of Theorem 7.1, we find $\frac{\sum_i |d_i|}{h_{\rm ex}} \to 0$, which means that the number of vortices is $o(h_{\rm ex})$. In other words, for energy-minimizers, vortices first appear for $\frac{h_{\rm ex}}{|\log \varepsilon|} \to \frac{1}{2|\underline{\xi_0}|}$, or $h_{\rm ex} \sim H^0_{c_1}$, and next to Λ (defined in (7.5)), and the vorticity mass is much smaller than $h_{\rm ex}$. The analysis of Chapter 7 does not give us the optimal number n of vortices nor the full asymptotic expansion of the first critical field. Thus, a more detailed study will be necessary in this regime $h_{\rm ex} \sim \frac{|\log \varepsilon|}{2|\underline{\xi_0}|}$, in which $n \ll h_{\rm ex}$. We will prove that the vortices, even though their number may be diverging, all concentrate around Λ (generically a single point) but that after a suitable blow-up, they tend to arrange in a uniform density on a subdomain of $\mathbb{R}^2$, in order to minimize a limiting interaction energy I defined on probability measures.

Many of the elements of the proof in this chapter will be useful in the following chapters.

9.1 Main Result

In this chapter, we assume for simplicity that Ω is a domain such that defining ξ_0 as in (7.2), the minimum of ξ_0 is achieved at a single point (recall Lemma 7.1) and that moreover $D^2\xi_0(p)$ is a positive definite quadratic form. This is the case if Ω is a ball or a convex set, for instance.

Throughout the chapter, we denote by p the unique point where ξ_0 achieves its minimum, by $\underline{\xi_0}$ its minimum value, and let $Q(x) = \langle D^2\xi_0(p)x, x\rangle$.

9.1.1 Motivation

The analysis which follows is best introduced by some formal calculations. Assume we are given solutions $\{(u_\varepsilon, A_\varepsilon)\}$ to the Ginzburg–Landau equations with an applied field h_{ex} which is of the order of $|\log\varepsilon|$. We drop the subscript ε for the clarity of notation. To (u, A) is associated the vorticity measure $\mu(u, A) = \operatorname{curl}(iu, \nabla_A u) + h$ and also, with the help of Theorem 4.1, a family of vortex balls with centers and degrees $\{(a_i, d_i)\}_i$, and total radius to be chosen later. We assume for simplicity that the degrees are all equal to $+1$ and let n denote the number of vortices. We also assume that every vortex ball has the same radius r. We wish to guess as precisely as possible where it is energetically favorable to place the vortices, if we know n to be small compared to h_{ex}.

Since (u, A) is a solution of the Ginzburg–Landau equations (GL), we know from Proposition 3.9 that $|u| \le 1$ and therefore, writing $j = (iu, \nabla_A u)$, we have $|\nabla_A u| \ge |j|$. On the other hand we have $-\nabla^\perp h = j$, where $h = \operatorname{curl} A$, thus

$$G_\varepsilon(u, A) \ge \frac{1}{2}\int_\Omega |\nabla h|^2 + |h - h_{\text{ex}}|^2 = \frac{1}{2}\|h - h_{\text{ex}}\|_{H^1}^2.$$

In this section we will make the assumption (which will be a posteriori justified) that minimizing the right-hand side or minimizing the left-hand side with respect to the number and/or positions of the vortices, is equivalent.

We decompose h as $h = h_{\text{ex}} h_0 + h_1$, where h_0 solves (7.1). The H^1 norm of $h - h_{\text{ex}}$ decomposes as

$$\|h - h_{\text{ex}}\|_{H^1}^2 = {h_{\text{ex}}}^2 \|h_0 - 1\|_{H^1}^2 + 2h_{\text{ex}}\langle h_1, h_0 - 1\rangle_{H^1} + \|h_1\|_{H^1}^2. \quad (9.1)$$

Taking the curl of the second Ginzburg–Landau equation we find $-\Delta h_1 + h_1 = \mu(u, A)$ and from the boundary conditions for h and h_0 we get $h_1 = 0$ on $\partial\Omega$. Integrating by parts the scalar product we then find, using the notation (7.2) and (7.3),

$$\frac{1}{2}\|h - h_{\text{ex}}\|_{H^1}^2 = {h_{\text{ex}}}^2 J_0 + h_{\text{ex}} \int_\Omega \xi_0 \mu(u, A) + \frac{1}{2}\int_\Omega |\nabla h_1|^2 + {h_1}^2. \quad (9.2)$$

To make the vortex positions appear, we recall that the vorticity $\mu(u, A)$ is close to $2\pi \sum_i \delta_{a_i}$ (see Theorem 6.1), and therefore h_1 is close to solving the "London equation" (as called in physics)

$$\begin{cases} -\Delta h_1 + h_1 = 2\pi \sum_i \delta_{a_i} & \text{in } \Omega \\ \qquad\qquad h_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Of course the true h_1 is smooth near the vortices, thus this approximation is valid only away from the vortices. We make the simplifying assumption that defining G_Ω as in (7.17), for $x \notin \cup_i B(a_i, r)$ we have $h_1(x) = 2\pi \sum_i G_\Omega(x, a_i)$. It is standard to check then that, using the notation (7.18),

$$\frac{1}{2} \int_{\Omega \setminus \cup_i B_i} |\nabla h_1|^2 + {h_1}^2 \approx \pi n \log \frac{1}{r} - \pi \sum_{i \neq j} \log |a_i - a_j| + \pi \sum_{i,j} S_\Omega(a_i, a_j).$$

As for the energy in each $B(a_i, r)$, in view of Theorem 4.1, we take it to be $\pi \log(r/\varepsilon)$. Together with (9.1), (9.2), where we replace $\mu(u, A)$ by $2\pi \sum_i \delta_{a_i}$, we find that $G_\varepsilon(u, A)$ can be approximated by

$${h_{\text{ex}}}^2 J_0 + \pi n |\log \varepsilon| + 2\pi h_{\text{ex}} \sum_i \xi_0(a_i) - \pi \sum_{i \neq j} \log |a_i - a_j| + \pi \sum_{i,j} S_\Omega(a_i, a_j).$$

We know that if n is small compared to h_{ex}, then the vortices will tend to concentrate near the minimum p of ξ_0. Therefore we are entitled to approximate $\xi_0(a_i)$ by $\underline{\xi_0} + \frac{1}{2} Q(a_i - p)$. We find

$$\begin{aligned} G_\varepsilon(u, A) \;\approx\; & {h_{\text{ex}}}^2 J_0 + \pi n |\log \varepsilon| + 2\pi h_{\text{ex}} n \underline{\xi_0} \\ & + \pi h_{\text{ex}} \sum_i Q(a_i - p) - \pi \sum_{i \neq j} \log |a_i - a_j| + \pi n^2 S_\Omega(p, p). \end{aligned} \tag{9.3}$$

What kind of information can we get from such considerations? Let ℓ denote the typical inter-vortex distance or rather the typical distance of a vortex to p. Three terms can be distinguished in the above sum. First the term

$${h_{\text{ex}}}^2 J_0 + \pi n |\log \varepsilon| + 2\pi h_{\text{ex}} n \underline{\xi_0} + \pi n^2 S_\Omega(p, p),$$

which depends only on the number, not the positions of the vortices. Then the term $\pi h_{\text{ex}} \sum_i Q(a_i - p)$ which favors concentration of the vortices near p, it is of the order of $n h_{\text{ex}} \ell^2$. Finally the term $-\pi \sum_{i \neq j} \log |a_i -$

$a_j|$ which is a repulsive term, is of the order of $-n^2 \log \ell$. To minimize G_ε we should then minimize $nh_{\text{ex}}\ell^2 - n^2 \log \ell$ and therefore choose

$$\ell = \sqrt{\frac{n}{h_{\text{ex}}}}.$$

Note that we are interested here in orders of magnitude, hence the constants are irrelevant. The natural next step is then to express (9.3) at a different scale, by letting $\tilde{a}_i = (a_i - p)/\ell$. We get

$$\begin{aligned} G_\varepsilon(u, A) \; &\approx \; {h_{\text{ex}}}^2 J_0 + \pi n |\log \varepsilon| + 2\pi n h_{\text{ex}} \underline{\xi_0} + \pi n^2 S_\Omega(p,p) \\ &\quad - \pi(n^2 - n) \log \ell + \pi n \sum_i Q(\tilde{a}_i) - \pi \sum_{i \neq j} \log |\tilde{a}_i - \tilde{a}_j|. \end{aligned}$$

This expansion is the sum of a term independent of the positions of the points,

$$f_\varepsilon(n) = {h_{\text{ex}}}^2 J_0 + \pi n \left(|\log \varepsilon| - 2h_{\text{ex}} |\underline{\xi_0}| \right) + \pi n^2 S_\Omega(p,p) + \pi(n^2 - n) \log \frac{1}{\ell}, \tag{9.4}$$

and a term best written in terms of the probability measure $\tilde{\mu} = \dfrac{1}{n} \sum_i \delta_{\tilde{a}_i}$ as $n^2 I(\tilde{\mu})$, where

$$I(\tilde{\mu}) = -\pi \iint \log |x - y| \, d\tilde{\mu}(x) \, d\tilde{\mu}(y) + \pi \int Q(x) \, d\tilde{\mu}(x).$$

9.1.2 Γ-Convergence in the Intermediate Regime

Throughout this chapter, we are given configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ to which we associate certain quantities that we define below.

Notation

Assume $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ satisfy, for some $\alpha \in (0,1)$,

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le \varepsilon^{\alpha - 1}, \quad C h_{\text{ex}} \le \varepsilon^{-\alpha}, \tag{9.5}$$

where $A'_\varepsilon = A_\varepsilon - h_{\text{ex}} \nabla^\perp h_0$ and $C > 0$ is a universal constant that we choose below.

From Theorem 4.1 applied to $(u_\varepsilon, A'_\varepsilon)$, we may construct vortex balls of radius $C\varepsilon^{\alpha/2}$, for some universal constant C. We define these to be the *small balls*, denoted by $\mathcal{B}' = \{B'_i\}_i$. Their centers and degrees are denoted by a'_i and d'_i, or more precisely d'_i is the degree of B'_i if $B'_i \subset \Omega_\varepsilon$ and $d'_i = 0$ otherwise. We let

$$r' = C\varepsilon^{\alpha/2}, \quad n' = \sum_i |d'_i|. \tag{9.6}$$

Under the hypothesis (9.5) and choosing the constant large enough, we have $r' < 1/\sqrt{h_{\text{ex}}}$, and therefore, using Theorem 4.2, we can grow the family of small balls $\mathcal{B}'$ into a family of *large balls* denoted by $\mathcal{B} = \{B_i\}_i$, of total radius $1/\sqrt{h_{\text{ex}}}$. We write a_i for the center of B_i and d_i for its degree, we also use the notation

$$r = \frac{1}{\sqrt{h_{\text{ex}}}}, \quad n = \sum_i |d_i|. \tag{9.7}$$

Note that r, n, r', n' all depend on ε. Also note that since every ball in $\mathcal{B}'$ is included in one of the balls in $\mathcal{B}$, Lemma 4.2 implies that $n' \geq n$.

The previous section motivates the following definitions. We let

$$\ell = \sqrt{\frac{n}{h_{\text{ex}}}} \tag{9.8}$$

and write

$$\begin{aligned} f_\varepsilon(n) = {h_{\text{ex}}}^2 J_0 + \pi n \log \frac{\ell}{\varepsilon} - 2\pi n h_{\text{ex}} |\underline{\xi_0}| \\ + \pi n^2 S_\Omega(p,p) + \pi n^2 \log \frac{1}{\ell}, \end{aligned} \tag{9.9}$$

$$f^0_\varepsilon(n) = \pi n \log \frac{\ell}{\varepsilon} + \pi n^2 S_\Omega(p,p) + \pi n^2 \log \frac{1}{\ell}. \tag{9.10}$$

Also, recalling that p denotes the unique point of minimum of ξ_0, we let φ be the blow-up centered at p for the scale ℓ, defined by

$$\varphi(x) = \frac{x - p}{\ell}. \tag{9.11}$$

If μ is a measure we will denote by $\tilde{\mu}$ its push-forward by the mapping φ, defined by $\tilde{\mu}(U) = \mu(\varphi^{-1}(U))$, and if x is a point, then we will let $\tilde{x} = \varphi(x)$.

Finally we denote by $\mathcal{P}$ the set of probability measures on $\mathbb{R}^2$ and for $\mu \in \mathcal{P}$ we let

$$\boxed{I(\mu) = -\pi \iint \log|x-y|\,d\mu(x)\,d\mu(y) + \pi \int Q(x)\,d\mu(x)}. \tag{9.12}$$

Results

We may now state the precise Γ-convergence result in the intermediate regime:

Theorem 9.1. (Γ-convergence in the intermediate regime, lower bound). *Assume that* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *satisfies* $F_\varepsilon(u_\varepsilon, A_\varepsilon) < \varepsilon^{-1/4}$ *and that* $h_{ex} < \varepsilon^{-1/8}$. *In particular* (9.5) *is satisfied with* $\alpha = 3/4$. *We also assume that* $1 \ll n \ll h_{ex}$ *as* $\varepsilon \to 0$, *that*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le f_\varepsilon(n) + Cn^2,$$

and we make one of the following two assumptions:

$$h_{ex} \le C|\log\varepsilon| \quad \text{or} \quad n' = n,$$

where n *and* n' *are defined by* (9.6)–(9.7). *Then the following holds.*

1. *There exists a probability measure* μ_* *such that, after extraction of a subsequence,*
$$\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n} \longrightarrow \quad \mu_*$$
as $\varepsilon \to 0$, *in the dual of* $C_c^{0,\gamma}(\mathbb{R}^2)$ *for some* $\gamma > 0$.

2. *As* $\varepsilon \to 0$,
$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge f_\varepsilon(n) + n^2 I(\mu_*) + o(n^2), \tag{9.13}$$
$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \ge f_\varepsilon^0(n) - n^2\pi \iint \log|x-y|\,d\mu_*(x)\,d\mu_*(y) + o(n^2), \tag{9.14}$$
where I *was defined in (9.12) and* f_ε *in (9.9).*

The corresponding upper bound also holds, with less restrictive assumptions on h_{ex}.

Proposition 9.1. *Given a probability measure μ with compact support in $\mathbb{R}^2$ such that $I(\mu) < \infty$ and given $1 \ll n_\varepsilon \ll h_{ex}(\varepsilon) \leq \varepsilon^{-\beta}$ with $\beta < 1$ as $\varepsilon \to 0$, there exists $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that $\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon} \to \mu$ in $(C_c^{0,\gamma}(\mathbb{R}^2))^*$ for every $\gamma > \beta/2$ and such that moreover*

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) = f^0_\varepsilon(n_\varepsilon) - n_\varepsilon^2 \pi \iint \log|x-y|\, d\mu(x)\, d\mu(y) + o(n_\varepsilon^2) \quad (9.15)$$

where $A'_\varepsilon = A_\varepsilon - h_{ex} \nabla^\perp h_0$, and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(n_\varepsilon) + n_\varepsilon^2 I(\mu) + o(n_\varepsilon^2). \quad (9.16)$$

We have therefore identified the right limiting object in this regime. It is the limit μ_* of the rescaled and normalized vorticity measures

$$\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon}.$$

To leading order, minimizing the energy of a configuration with n vortices reduces to minimizing I. We describe below some of what is known about this minimization problem, and then give precisions on the behavior of $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ as $\varepsilon \to 0$.

Minimization of I

The solution to the minimization of I is known. It falls into the more general problem of minimizing functionals of the type

$$\iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y) + \int Q(x)\, d\mu(x)$$

over probability measures, when e^{-Q} decreases fast enough at infinity. We cite the following result from [163]:

Proposition 9.2 (See [163]). *$\inf_{\mu \in \mathcal{P}} I(\mu)$ is finite and there is a unique minimizer μ_0, which has compact support. It is uniquely characterized by the fact that there exists a constant F such that*

$$U^{\mu_0} + \frac{1}{2} Q = F \quad \textit{quasi-everywhere on Supp } \mu_0$$
$$U^{\mu_0} + \frac{1}{2} Q \geq F \quad \textit{quasi-everywhere in } \mathbb{R}^2$$

where $U^{\mu_0}(x) = \int_{\mathbb{R}^2} -\log|x-y|\, d\mu_0(y)$.

If Q is a positive definite quadratic form, then μ_0 is a measure supported on a compact set of $\mathbb{R}^2$ of constant density $\frac{1}{2}\Delta Q$ (the Laplacian of Q).

Corollary 9.1. *Under the hypotheses of Theorem 9.1,*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) - f_\varepsilon(n) \geq \pi n^2 I_0 + o(n^2) \tag{9.17}$$

where

$$I_0 = \min_{\mu \in \mathcal{P}} I(\mu). \tag{9.18}$$

Moreover, if $n \geq 1$ and if there is equality in (9.17), then we must have

$$\frac{\tilde{\mu}(u_\varepsilon, A_\varepsilon)}{2\pi n} \rightharpoonup \mu_0 \quad \text{in } (C_c^{0,\gamma}(\mathbb{R}^2))^*, \ \forall \gamma > 0$$

where μ_0 is the minimizer of (9.18).

Proof. The result is immediate and follows from the uniqueness of the minimizer of I_0. □

Since Q is a quadratic function in our case, we also know from Proposition 9.2 that the minimizer μ_0 has a constant density $\frac{1}{2}\Delta Q$.

We now turn to the proofs of these main results.

9.2 Upper Bound: Proof of Proposition 9.1

Let us assume that the support of μ is included in $B(0,R)$. The fact that $I(\mu) < \infty$ and $\mu \geq 0$ implies that $\mu \in H^{-1}(B(0,R))$. Indeed, introducing $S_{B(0,R)}$ and $G_{B(0,R)}$ with the notation (7.17)–(7.18), we have

$$\iint G_{B(0,R)}(x,y)\, d\mu(x)\, d\mu(y) = \\ \frac{1}{2\pi}\iint S_{B(0,R)}(x,y)\, d\mu(x)\, d\mu(y) - \frac{1}{2\pi}\iint \log|x-y|\, d\mu(x)\, d\mu(y).$$

The first term in the right-hand side is finite because $S_{B(0,R)}$ is a continuous function, and the second term by finiteness of $I(\mu)$. We deduce that $\iint G_{B(0,R)}\,d\mu\,d\mu < \infty$ and hence that $\mu \in H^{-1}(B(0,R))$. We may then apply Proposition 7.4 in $B(0,R)$ with ε replaced by

$$\tilde{\varepsilon} = \frac{\varepsilon}{\ell},$$

(remark that $\tilde{\varepsilon} \ll \varepsilon\sqrt{h_{\rm ex}} \le C$) we obtain the existence of n_ε points $b_i^\varepsilon \in B(0,R)$ such that

$$\frac{1}{n_\varepsilon}\sum_i \tilde{\mu}_\varepsilon^i \quad \to \quad 2\pi\mu \tag{9.19}$$

in the weak sense of measures, and

$$\lim_{\varepsilon\to 0} -\frac{1}{{n_\varepsilon}^2}\sum_{i\neq j}\iint \log|x-y|\,d\tilde{\mu}_\varepsilon^i(x)\,d\tilde{\mu}_\varepsilon^j(y) = -4\pi^2\iint \log|x-y|\,d\mu(x)\,d\mu(y), \tag{9.20}$$

where the measures $\tilde{\mu}_\varepsilon^i$ are the uniform measures on $\partial B(b_i^\varepsilon, \tilde{\varepsilon})$ and of mass 2π (hence also $\frac{1}{n_\varepsilon}\sum_i \delta_{b_i^\varepsilon} \to \mu$). Let us now rescale and consider $a_i^\varepsilon = \varphi^{-1}(b_i^\varepsilon)$ (where φ is defined in (9.11)) and μ_ε^i the pull-back under φ of the measures $\tilde{\mu}_\varepsilon^i$, i.e., uniform measures on $\partial B(a_i^\varepsilon, \varepsilon)$. We may apply Proposition 7.3 to those $\{a_i^\varepsilon\}$. We get a configuration $(u_\varepsilon, A_\varepsilon)$ for which (7.23), (7.24) and (7.25) hold.

But, clearly $\frac{1}{n_\varepsilon}\sum_i \mu_\varepsilon^i \rightharpoonup 2\pi\delta_p$, the Dirac mass at p, and thus, by continuity of S_Ω,

$$\frac{1}{4\pi}\sum_{i,j}\iint S_\Omega(x,y)\,d\mu_\varepsilon^i(x)\,d\mu_\varepsilon^j(y) = \pi n_\varepsilon^2 S_\Omega(p,p) + o(n_\varepsilon^2).$$

On the other hand, using the change of variables $y = \varphi(x)$ and (9.20), we find

$$-\frac{1}{4\pi}\sum_{i\neq j}\iint \log|x-y|\,d\mu_\varepsilon^i(x)\,d\mu_\varepsilon^j(y) = \pi(n_\varepsilon^2 - n_\varepsilon)\log\frac{1}{\ell} - \pi n_\varepsilon^2\iint \log|x-y|\,d\mu(x)\,d\mu(y) + o(n_\varepsilon^2).$$

Finally, inserting these relations into (7.23), we get (9.15). Next, observe that $n_\varepsilon \gg 1$ and $h_{\rm ex} \le \varepsilon^{-1}$ thus $\ell^2 = \frac{n_\varepsilon}{h_{\rm ex}} \gg \frac{1}{h_{\rm ex}} \gg \varepsilon$. We deduce that $\log \frac{1}{\ell} \le |\log \varepsilon|$, and that $f_\varepsilon^0(n_\varepsilon) \le O(n_\varepsilon^2 |\log \varepsilon|)$, so $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le O(n_\varepsilon^2 |\log \varepsilon|)$. Inserting that into (7.24) and using $h_{\mathrm{ex}} \le \varepsilon^{-\beta}$ and (9.19) we get that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = F_\varepsilon(u_\varepsilon, A'_\varepsilon) + {h_{\mathrm{ex}}}^2 J_0 + \sum_i h_{\mathrm{ex}} \int \xi_0(x)\, d\mu_\varepsilon^i(x) + o(n_\varepsilon^2). \tag{9.21}$$

But by definition of the μ_ε^i's and smoothness of ξ_0, we easily have that

$$\sum_i h_{\mathrm{ex}} \int \xi_0(x)\, d\mu_\varepsilon^i(x) = 2\pi h_{\mathrm{ex}} \sum_i \xi_0(a_i^\varepsilon) + O(n_\varepsilon h_{\mathrm{ex}} \varepsilon).$$

Then, using a Taylor expansion at p, since $a_i^\varepsilon = p + \ell b_i^\varepsilon$, $\nabla \xi_0(p) = 0$ and Q is the Hessian of ξ_0 there, we have

$$\begin{aligned} 2\pi h_{\mathrm{ex}} \sum_i \xi_0(a_i^\varepsilon) &= 2\pi h_{\mathrm{ex}} \sum_i \left(\xi_0(p) + \frac{\ell^2}{2} Q(b_i^\varepsilon) + o\left(\ell^2\right) \right) \\ &= -2\pi n_\varepsilon h_{\mathrm{ex}} |\underline{\xi_0}| + \pi n_\varepsilon \sum_i Q(b_i^\varepsilon) + o(n_\varepsilon^2) \\ &= -2\pi n_\varepsilon h_{\mathrm{ex}} |\underline{\xi_0}| + \pi n_\varepsilon^2 \int Q(x)\, d\mu(x) + o(n_\varepsilon^2). \end{aligned}$$

Inserting this into (9.21), then using (9.15) and the definition of f_ε and I, we conclude that (9.16) holds.

We now turn to the convergence of $\tilde{\mu}(u_\varepsilon, A_\varepsilon)$. Using (7.25) and rescaling, we find

$$\begin{aligned} &\left\| \frac{1}{n_\varepsilon} \left(\tilde{\mu}(u_\varepsilon, A_\varepsilon) - \sum_{i=1}^{n_\varepsilon} \tilde{\mu}_\varepsilon^i \right) \right\|_{(C^{0,\gamma}(\mathbb{R}^2))^*} \\ &\qquad \le \ell^{-\gamma} \left\| \frac{1}{n_\varepsilon} \left(\mu(u_\varepsilon, A_\varepsilon) - \sum_{i=1}^{n_\varepsilon} \mu_\varepsilon^i \right) \right\|_{(C^{0,\gamma}(\Omega))^*} \\ &\qquad \le C \ell^{-\gamma} \varepsilon^\gamma \left(1 + \varepsilon h_{\mathrm{ex}} + \sqrt{\frac{F_\varepsilon(u_\varepsilon, A'_\varepsilon)}{n_\varepsilon}} \right). \end{aligned} \tag{9.22}$$

Using again $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le O(n_\varepsilon^2|\log\varepsilon|)$, we have

$$\ell^{-\gamma}\varepsilon^\gamma\left(1+\varepsilon h_{\rm ex}+\sqrt{\frac{F_\varepsilon(u_\varepsilon,A'_\varepsilon)}{n_\varepsilon}}\right) \le Cn_\varepsilon^{-\gamma/2}{h_{\rm ex}}^{\gamma/2}\varepsilon^\gamma(1+n_\varepsilon^{1/2}|\log\varepsilon|^{1/2})$$
$$\le o(1)+O\left(n_\varepsilon^{(1-\gamma)/2}{h_{\rm ex}}^{\gamma/2}\varepsilon^\gamma|\log\varepsilon|^{1/2}\right).$$

Next, we insert $n_\varepsilon \ll h_{\rm ex} \le \varepsilon^{-\beta}$ and find

$$\ell^{-\gamma}\varepsilon^\gamma\left(1+\varepsilon h_{\rm ex}+\sqrt{\frac{F_\varepsilon(u_\varepsilon,A'_\varepsilon)}{n_\varepsilon}}\right) \le o(1)+o({h_{\rm ex}}^{1/2}\varepsilon^\gamma)$$
$$\le o(1)+o(\varepsilon^{\gamma-\beta/2}|\log\varepsilon|^{1/2})$$

which is $o(1)$ as soon as $\gamma > \beta/2$. Combining this with (9.22) and (9.19), we get the stated convergence for $\dfrac{\tilde\mu(u_\varepsilon, A_\varepsilon)}{2\pi n_\varepsilon}$.

The rest of the chapter is devoted to proving Theorem 9.1.

9.3 Proof of Theorem 9.1

Before presenting the proof, let us explain the main steps and ingredients.
The first step is to split the energy in the following way, which corresponds to (9.2) and Lemma 7.3. We let $A' = A - h_{\rm ex}\nabla^\perp h_0$ and show that

$$G_\varepsilon(u,A) = {h_{\rm ex}}^2 J_0 + F_\varepsilon(u,A') + 2\pi h_{\rm ex}\sum_i d_i\xi_0(a_i) + o(1). \qquad (9.23)$$

Then, we recover as a lower bound all the energy contributions found in the upper bound (9.16), beginning with the terms which constitute $f_\varepsilon(n)$. First the main order ones: $\pi n\log(\ell/\varepsilon)$ coming from the vortex-core energy, and $-2\pi n h_{\rm ex}|\underline{\xi_0}|$ the main order term of $2\pi h_{\rm ex}\sum_i d_i\xi_0(a_i)$ coming from the interaction with the magnetic field; then the term $\pi n^2\log(1/\ell)$ will come from a more delicate lower bound on the energy on an annulus centered at p (Proposition 9.4).

Finally, when all these terms are obtained and yield $f_\varepsilon(n)$, comparing with the upper bound proves that whatever remains is of the lower order

n^2. This allows us to get compactness and pass to the limit in the remaining terms, like in a renormalized energy procedure, bounding them from below by the limiting ("renormalized energy") I.

In this process, it is crucial to locate the energy contributions and retrieve them where they are, because the terms have different orders of magnitude. More precisely, the vortex-core energy will come as usual from the energy in the vortex-balls constructed to be large enough but still small (total radius ${h_{\text{ex}}}^{-1/2}$). Then, we will split Ω into three regions: a ball $B(p, K\ell)$, an annulus $\mathcal{A} = B(p, \delta)\backslash B(p, K\ell)$, and the complement of $B(p, \delta)$ (see Fig. 9.1).

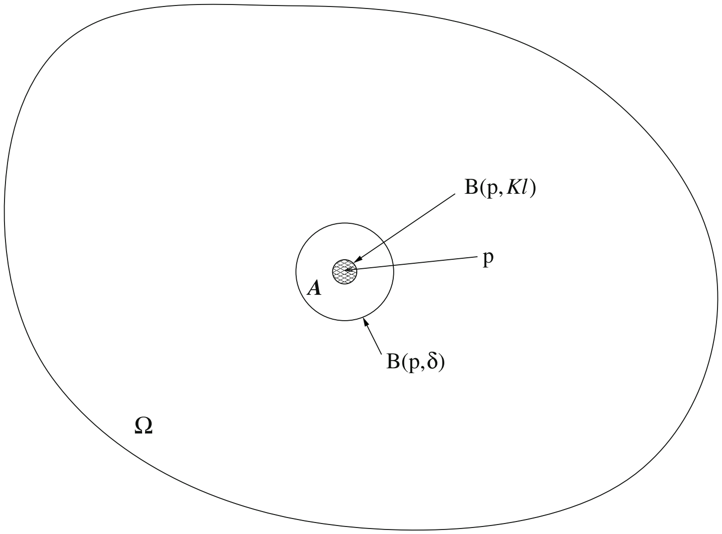

Figure 9.1: The annulus.

Essentially, the contribution of the annulus $\mathcal{A}$ gives the $\pi n^2 \log(1/\ell)$ term, and ensures that almost all the vortices remain confined in the inner disc $B(p, K\ell)$ if K (independent of ε) is large enough. Then, the energy of $I(\mu)$ will come from the contributions of the complement of $\mathcal{A}$.

One of the first technical difficulties to overcome is that the splitting (9.23) is only valid if the vortices a_i's correspond to small enough balls. On the other hand, to retrieve the total energy of the vortex cores, we need larger balls. A first step in the analysis will thus consist in going from small to large balls.

9.3.1 Energy-Splitting Lower Bound

In this section we prove the following:

Proposition 9.3. *Under the hypothesis* (9.5) *and using the above notation, there exist points* $\{b_i\}_i$ *such that* $b_i \in B_i$ *for every* i *and, letting* $\nu = \sum_i d_i \delta_{b_i}$,

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq {h_{ex}}^2 J_0 + 2\pi h_{ex} \int \xi_0 \, d\nu + F_\varepsilon(u_\varepsilon, A'_\varepsilon) \\ - C(n'-n) r h_{ex} - C h_{ex} \varepsilon^{\frac{3\alpha}{2}-1} - C {h_{ex}}^2 \varepsilon^\alpha, \qquad (9.24)$$

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \geq \pi n \log \frac{r}{n\varepsilon} + F_\varepsilon(u_\varepsilon, A'_\varepsilon, \Omega \setminus \mathcal{B}) + \frac{1-r^2}{2} \int_{\mathcal{B}} (\operatorname{curl} A')^2 \\ + \pi \frac{\alpha}{2}(n'-n)|\log \varepsilon| - Cn. \qquad (9.25)$$

Note that in the above we have abused the notation by writing $F_\varepsilon(u_\varepsilon, A'_\varepsilon, \Omega \setminus \mathcal{B})$ instead of $F_\varepsilon(u_\varepsilon, A'_\varepsilon, \Omega \setminus \cup_i B_i)$ and $\int_{\mathcal{B}}$ instead of $\int_{\cup_i B_i}$.

Using the energy-splitting lemma, Lemma 7.3, the proof of (9.24) consists in approximating $\mu(u, A')$ by the measure $2\pi \sum_i d_i \delta_{b_i}$. Unfortunately, $r = 1/\sqrt{h_{\rm ex}}$ is too large for Theorem 6.1 to apply. We must then play with small and large balls. We first need the following:

Lemma 9.1. *We have, assuming* (9.5),

$$\frac{1}{2} \int_{\mathcal{B}} |\nabla_{A'} u|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2 + r^2 (\operatorname{curl} A')^2 \\ \geq \pi \left(n \log \frac{r}{n\varepsilon} + \frac{\alpha}{2}(n'-n)|\log \varepsilon| \right) - Cn, \quad (9.26)$$

where we have dropped the subscript ε *for* u *and* A'.

Proof. In this proof, we add up the lower bounds found on the small balls of $\mathcal{B}'$ to the lower bound in the large annuli $\mathcal{B} \backslash \mathcal{B}'$ (with an abuse of notation). From Theorem 4.1 we have, with the notation of (9.6):

$$\frac{1}{2} \int_{\mathcal{B}'} |\nabla_{A'} u|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2 + r'^2 (\operatorname{curl} A')^2 \\ \geq \pi n' \left(\log \frac{r'}{n'\varepsilon} - C \right). \quad (9.27)$$

On the other hand, applying Proposition 4.3 in Ω_ε (see (4.2)) to $v = u/|u|$ with $\mathcal{B}_0 = \mathcal{B}'$ and final radius r we get, restricting every integral below to Ω_ε,

$$\frac{1}{2}\int_{\mathcal{B}\setminus\mathcal{B}'} |\nabla_{A'}v|^2 + r(r-r')(\operatorname{curl} A')^2 \geq \pi n \log \frac{r}{2r'}. \tag{9.28}$$

Now recall that from Theorem 4.1 we have $|u| \geq 1 - \varepsilon^{\alpha/4}$ in $\Omega_\varepsilon \setminus \mathcal{B}'$ and therefore

$$|\nabla_{A'}u|^2 \geq |u|^2 |\nabla_{A'}v|^2 \geq (1 - 2\varepsilon^{\alpha/4})|\nabla_{A'}v|^2$$

there. Therefore, multiplying (9.28) by $(1 - 2\varepsilon^{\alpha/4})$ we get

$$\frac{1}{2}\int_{\mathcal{B}\setminus\mathcal{B}'} |\nabla_{A'}u|^2 + r(r-r')(\operatorname{curl} A')^2 \geq \pi n \log \frac{r}{r'} - Cn, \tag{9.29}$$

where we have used the fact that since $r/r' < C\varepsilon^{-\alpha/2}$, the quantity $\varepsilon^{\alpha/4}\log(r/r')$ is bounded by a constant.

Adding (9.27) and (9.29) yields

$$\frac{1}{2}\int_{\mathcal{B}} |\nabla_{A'}u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 + r^2(\operatorname{curl} A')^2$$
$$\geq \pi\left(n\log\frac{r}{n\varepsilon} + (n'-n)\log\frac{r'}{\varepsilon} - n'\log n' + n\log n - Cn'\right). \tag{9.30}$$

Now,

$$n'\log n' - n\log n = \int_n^{n'} (\log t + 1)\,dt \leq (n'-n)\left(\log n' + 1\right), \tag{9.31}$$

and from Theorem 4.1 we know that $n' \leq CF_\varepsilon(u, A')/(\alpha|\log\varepsilon|) \leq C\varepsilon^{\alpha-1}$. It follows that $\log n' \leq (1-\alpha)|\log\varepsilon| + C$. Inserting into (9.31) and using the fact that $r'/\varepsilon = C\varepsilon^{\frac{\alpha}{2}-1}$ we find

$$(n'-n)\log\frac{r'}{\varepsilon} - n'\log n' + n\log n - Cn' \geq (n'-n)\left(\frac{\alpha}{2}|\log\varepsilon| - C\right) - Cn'.$$

Thus, for ε small enough, the above allows us to write (9.30) as (9.26), hereby proving the lemma. □

We also have:

Lemma 9.2. *Assuming* (9.5)*, we may choose points in the large balls, i.e.,* $\{b_i\}_i$ *such that* $b_i \in B_i$*, such that letting*

$$\nu = \sum_i d_i \delta_{b_i}, \quad \nu' = \sum_j d'_j \delta_{a'_j}$$

be respectively the measures relative to the large (ν) *and small* (ν') *balls, we have*

$$\int \xi_0 \, d\nu' = \int \xi_0 \, d\nu - R_1, \tag{9.32}$$

where

$$R_1 \le C(n' - n)r.$$

Proof. What we are trying to do is to bound from above $\int \xi_0 \, d(\nu - \nu')$. To get rid of the problem of balls intersecting the boundary, we define ν_1 as the part of ν' which corresponds to small balls which are included in a large ball that does not intersect the boundary, and ν_2 as the remaining part. More precisely, we let J_1 be the set of indices j such that for some i we have $B'_j \subset B_i$ with $B_i \subset \Omega_\varepsilon$ and we define

$$\nu_1 = \sum_{j \in J_1} d'_j \delta_{a'_j}, \quad \nu_2 = \nu' - \nu_1.$$

We begin by showing that ν_2 can be ignored. Indeed if $j \notin J_1$, then this means that the point a'_j is at a distance smaller than $r + \varepsilon$ from the boundary of Ω hence $|\xi_0(a'_j)| \le C(r + \varepsilon)$. It follows that

$$\left| \int \xi_0 \, d\nu_2 \right| \le C(r + \varepsilon)(n' - n) \le Cr(n' - n), \tag{9.33}$$

where we have used the fact that $r \ge \varepsilon$ and we have bounded $\sum_{j \notin J_1} |d'_j|$ by $n' - n$.

From (9.33), we are reduced to proving the lemma, but with ν' replaced by ν_1, which we do now. First we choose the points b_i. We have

$$\int \xi_0 \, d(\nu - \nu_1) = \sum_i d_i \xi_0(b_i) - \sum_{j \in J_1} d'_j \xi_0(a'_j).$$

But for every i such that $B_i \subset \Omega_\varepsilon$ we have $d_i = \sum d'_j$, where the sum runs over the indices j such that $B'_j \subset B_i$, while if $B_i \not\subset \Omega_\varepsilon$, then $d_i = 0$. Thus we may rewrite the above as

$$\int \xi_0 \, d(\nu - \nu_1) = \sum_{B_i \subset \Omega_\varepsilon} \sum_{B'_j \subset B_i} d'_j \left(\xi_0(b_i) - \xi_0(a'_j)\right).$$

This sum is made the smallest by choosing $b_i \in B_i$ such that

$$\xi_0(b_i) = \begin{cases} \min_{B_i} \xi_0 & \text{if } d_i \geq 0 \\ \max_{B_i} \xi_0 & \text{otherwise.} \end{cases} \tag{9.34}$$

Then we have $d'_j \Big(\xi_0(b_i) - \xi_0(a'_j)\Big) \leq 0$ whenever $B'_j \subset B_i$ and $d'_j d_i \geq 0$. Therefore, assuming from now on (9.34),

$$\int \xi_0 \, d(\nu - \nu_1) \leq \sum_{B_i \subset \Omega_\varepsilon} \sum_{\substack{B'_j \subset B_i \\ d'_j d_i < 0}} d'_j \left(\xi_0(b_i) - \xi_0(a'_j)\right). \tag{9.35}$$

Now we observe that

$$\sum_{B_i \subset \Omega_\varepsilon} \sum_{\substack{B'_j \subset B_i \\ d'_j d_i < 0}} |d'_j| = \frac{1}{2} \left(\sum_{j \in J_1} |d'_j| - \sum_i |d_i| \right) \leq \frac{n' - n}{2}, \tag{9.36}$$

while for every j such that $B'_j \subset B_i$, since a'_j and b_i both belong to B_i which has radius less than r,

$$\left|\xi_0(b_i) - \xi_0(a'_j)\right| \leq Cr \|\nabla \xi_0\|_\infty.$$

Inserting the above and (9.36) into (9.35) we get

$$\int \xi_0 \, d(\nu - \nu_1) \leq Cr(n' - n) \|\nabla \xi_0\|_\infty.$$

Together with (9.33), this proves the lemma. □

Proof of Proposition 9.3. Let us write in shorthand $\mu = \mu(u, A')$ and $h' = \operatorname{curl} A'$.

In view of (7.22), and noticing that $\varepsilon F_\varepsilon(u, A') \le C\varepsilon^\alpha$, proving (9.24) reduces to proving that

$$\int_\Omega \xi_0 \mu \ge \int \xi_0 \, d\nu - C(n' - n)r - C\varepsilon^{\frac{3\alpha}{2}-1},$$

which from Lemma 9.2 in turn reduces to proving that

$$\int_\Omega \xi_0 \mu \ge \int \xi_0 \, d\nu' - C\varepsilon^{\frac{3\alpha}{2}-1}.$$

This last inequality follows from Theorem 6.1, by noticing that $r' F_\varepsilon(u, A') \le C\varepsilon^{\frac{\alpha}{2}+\alpha-1}$.

It remains to prove (9.25), but this is a direct consequence of Lemma 9.1, if we write $F_\varepsilon(u, A') = F_\varepsilon(u, A', \Omega \setminus \mathcal{B}) + F_\varepsilon(u, A', \mathcal{B})$ and further split this expression by writing $F_\varepsilon(u, A', \mathcal{B})$ as

$$\frac{1}{2}\int_{\mathcal{B}} \left(|\nabla_{A'} u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2 + r^2 h'^2 \right) + \frac{1}{2}(1 - r^2)\int_{\mathcal{B}} h'^2. \qquad \square$$

9.3.2 Lower Bound on the Annulus

The next step, after (9.24)–(9.25) are obtained, is to retrieve from it the remaining terms in $f_\varepsilon(n)$. Roughly speaking, these will come from a lower bound of $F_\varepsilon(u, A', \mathcal{A})$, where $\mathcal{A}$ is a carefully chosen annulus. Recall that p denotes the unique point where ξ_0 achieves its minimum.

We still assume that (9.5) is satisfied and we use the same notation as above for the small balls, large balls, and related quantities. We also denote by $\{b_i\}_i$ points chosen inside the large balls, i.e., $b_i \in B_i$ for every i.

Recall that n is the sum of the absolute values of the degrees of the large balls. Given arbitrary positive numbers K, δ, if the length ℓ as defined by (9.8) is small enough, then $K\ell < \delta$ and we may define our annulus $\mathcal{A}$ as follows (see Fig. 9.1).

$$\mathcal{A} = B(p, r_1) \setminus B(p, r_0), \quad r_0 = K\ell, \quad r_1 = \delta. \tag{9.37}$$

We will assume that $K > 1$ and that δ is small enough so that $\mathcal{A} \subset \Omega_\varepsilon$. We insist that K and δ are chosen independent of ε, whereas n, h_{ex} and therefore ℓ may or may not depend on ε, the latter being useful in later chapters. We will also sometimes write in shorthand $B_1 = B(p, r_1)$ and $B_0 = B(p, r_0)$.

It will also be useful to define the function $D : [r_0, r_1] \to \mathbb{Z}$ by

$$D(t) = \sum_{|b_i - p| \le t} d_i. \tag{9.38}$$

Note that if t is such that $\partial B(p,t)$ does not intersect the large balls, then

$$D(t) = \deg\left(u/|u|, \partial B(p,t)\right).$$

Finally we let

$$D^- = \sum_{d_i<0} |d_i|, \quad D_e = \sum_{\substack{d_i>0 \\ b_i \notin B(p,\delta)}} d_i. \tag{9.39}$$

We bound from below the contribution of the energy in the annulus $\mathcal{A}$, using the method we had introduced in [171] which consists in integrating over circles centered at p (the core of the idea is in Lemma 9.4). Since the degree of u on the annulus will be shown to be approximately constant equal to n, this will yield a lower bound of the free-energy in $\pi n^2 \log \frac{r_1}{r_0}$ up to error terms.

Proposition 9.4. *Assume* (9.5) *is satisfied. There exist positive numbers K_0, δ_0 depending on Ω such that if $K \ge K_0$, $\delta \le \delta_0$, and if ℓ is small enough depending on K, δ, Ω, letting $\nu = \sum_i d_i \delta_{b_i}$, we have*

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}u|^2 + \frac{1}{4}\int_{B(p,\delta)} (\operatorname{curl} A')^2 + 2\pi h_{ex}\int \xi_0\, d\nu \\ \ge \pi n^2 \log\frac{\delta}{K\ell} + 2\pi n h_{ex}\underline{\xi_0} \\ + 2\pi h_{ex}\sum_{\substack{b_i\in B(p,K\ell)\\ d_i>0}} d_i\left(\xi_0(b_i) - \underline{\xi_0}\right) - \pi n^2\delta^2 - \pi\frac{n^{3/2}}{K} + o(n^2).\end{aligned} \tag{9.40}$$

Moreover, if the difference between the left-hand side and the right-hand side is less than Mn^2, then D^- and D_e are bounded by Cn^2/h_{ex}, and for any $t \in [r_0, r_1]$,

$$\left|\frac{D(t)-n}{n}\right| \le C\left(\frac{\ell^2}{t^2} + \ell^2\right). \tag{9.41}$$

In this case we also have

$$\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}u|^2 + \frac{1}{4}\int_{B_1}(\operatorname{curl} A')^2 \geq \pi n^2 \log\frac{\delta}{K\ell} - Cn^2\left(\delta^2 + \frac{1}{K^2} + o(1)\right). \quad (9.42)$$

In the above, C depends on M, Ω, δ, K.

We begin with the following:

Lemma 9.3. *Under the same hypotheses as above,*

$$\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}u|^2 + \frac{1}{4}\int_{B_1}(\operatorname{curl} A')^2 \geq \pi\int_{r_0}^{r_1}\frac{D^2(t)}{t}\,dt - \pi n^2\delta^2 - \pi\frac{n^{3/2}}{K} - Cn^2\varepsilon^{\alpha/4}\log\frac{r_1}{r_0}. \quad (9.43)$$

Proof. Let $T = \{t \in (r_0, r_1) \mid \partial B(p,t) \cap \mathcal{B} \neq \varnothing\}$. Then the Lebesgue measure of T, denoted by $|T|$, is less than twice the total radius of the balls, i.e., $|T| \leq 2r$, where we recall that $r = 1/\sqrt{h_{\mathrm{ex}}}$. Moreover for any $t \notin T$, Lemma 4.4 applied with $\lambda = 1/(2\delta)$ yields

$$\frac{1}{2}\int_{\partial B(p,t)} |\nabla_{A'}v|^2 + \frac{1}{4\delta}\int_{B(p,t)} |\operatorname{curl} A'|^2 \geq \pi\frac{D^2(t)}{t}\frac{1}{1+t\delta} \geq \pi D^2(t)\left(\frac{1}{t} - \delta\right),$$

where $v = u/|u|$. Integrating with respect to $t \in (r_0, r_1) \setminus T$ we find

$$\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}v|^2 + \frac{1}{4}\int_{B_1} |\operatorname{curl} A'|^2 \geq \int_{r_0}^{r_1} \pi D^2(t)\left(\frac{1}{t} - \delta\right) dt - \int_T \pi D^2(t)\left(\frac{1}{t} - \delta\right) dt.$$

The integral over T can be estimated by bounding $|D|$ above by n, and noting that since $|T| \leq r$,

$$\int_T \frac{n^2}{t}\,dt \leq \int_{r_0}^{r_0+r}\frac{n^2}{t}\,dt = n^2\log\left(1 + \frac{r}{r_0}\right).$$

This yields

$$\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}v|^2 + \frac{1}{4}\int_{B_1} |\operatorname{curl} A'|^2 \geq \int_{r_0}^{r_1} \pi\frac{D^2(t)}{t}\,dt - \pi n^2\left(\delta^2 + \log\left(1+\frac{r}{r_0}\right)\right)$$
$$\geq \int_{r_0}^{r_1} \pi\frac{D^2(t)}{t}\,dt - \pi n^2\delta^2 - \pi\frac{n^{3/2}}{K},$$

where we have used the fact that $n^2 r/r_0 = n^{3/2}/K$, which follows from (9.37), (9.8) and (9.7).

Now we use again the fact that in $\mathcal{A}\setminus\mathcal{B}$, we have $|u| \geq 1-\varepsilon^{\alpha/4}$, and therefore

$$|\nabla_{A'}u|^2 \geq |u|^2|\nabla_{A'}v|^2 \geq (1-2\varepsilon^{\alpha/4})|\nabla_A v|^2.$$

This implies using the above that

$$\frac{1}{2}\int_{\mathcal{A}\setminus\mathcal{B}} |\nabla_{A'}u|^2 + \frac{1}{4}\int_{B_1} (\operatorname{curl} A')^2$$
$$\geq \pi\int_{r_0}^{r_1} \frac{D^2(t)}{t}\,dt - \pi n^2\delta^2 - \pi\frac{n^{3/2}}{K} - Cn^2\varepsilon^{\alpha/4}\log\frac{r_1}{r_0},$$

and therefore (9.43) is proved. □

We now estimate the right-hand side of (9.43).

Lemma 9.4. *Assume* (9.5) *is satisfied. There exist positive numbers* K_0, δ_0 *depending only on* Ω *such that if* $\delta \leq \delta_0$, $K \geq K_0$, *and* ℓ *is small enough depending on* K, δ, Ω, *then*

$$\pi\int_{r_0}^{r_1} \frac{D^2(t)}{t}\,dt + 2\pi h_{ex}\sum_i d_i\xi_0(b_i) \geq$$
$$\pi n^2\log\frac{r_1}{r_0} + 2\pi n h_{ex}\underline{\xi_0} + 2\pi h_{ex}\sum_{\substack{b_i\in B_0\\ d_i>0}} d_i\left(\xi_0(b_i) - \underline{\xi_0}\right). \quad (9.44)$$

Moreover, if the difference between the left-hand side and the right-hand side is $O(n^2)$, *then* D^- *and* D_e *are* $O\left(n^2/h_{ex}\right)$ *and for any* $t \in [r_0, r_1]$,

$$\left|\frac{D(t)-n}{n}\right| \leq C\left(\frac{\ell^2}{t^2}+\ell^2\right). \tag{9.45}$$

Proof. First we write $D^2 - n^2 = (n-D)^2 + 2n(D-n) \geq 2n(D-n)$. Then

$$\int_{r_0}^{r_1} \frac{D^2(t)}{t}\,dt - n^2 \log\frac{r_1}{r_0} = \int_{r_0}^{r_1} \frac{D^2(t)-n^2}{t}\,dt \geq 2n \int_{r_0}^{r_1} \frac{D(t)-n}{t}\,dt. \tag{9.46}$$

If we write $r_i = |b_i - p|$, we have $D(t) = \sum_{r_i \leq t} d_i$ while $n = \sum_i |d_i|$. Therefore, letting $\bar{r}_i = \max(r_0, \min(r_1, r_i))$,

$$\int_{r_0}^{r_1} \frac{D(t)-n}{t}\,dt = \sum_i \left(d_i \log\frac{r_1}{\bar{r}_i} - |d_i| \log\frac{r_1}{r_0}\right). \tag{9.47}$$

We now partition the set of indices for which $d_i \neq 0$ into the following sets.

$$I^- = \{i \mid d_i < 0\}, \qquad I_e = \{i \mid d_i > 0, r_i \geq r_1\},$$
$$I_0 = \{i \mid d_i > 0, r_i \leq r_0\}, \qquad I_{\mathcal{A}} = \{i \mid d_i > 0, r_0 < r_i < r_1\}.$$

Then, letting

$$\Delta = \pi \int_{r_0}^{r_1} \frac{D^2(t)}{t}\,dt - \pi n^2 \log\frac{r_1}{r_0} + 2\pi h_{\mathrm{ex}}\left(\sum_i d_i \xi_0(b_i) - n\underline{\xi_0}\right)$$

we have using (9.46), (9.47) and the fact that ξ_0 is a negative function

$$\begin{aligned}\frac{\Delta}{2\pi} \geq \sum_{i\in I_0} h_{\mathrm{ex}} d_i (\xi_0(b_i) - \underline{\xi_0}) + \sum_{i\in I^-} |d_i| \left(-2n\log\frac{r_1}{r_0} - h_{\mathrm{ex}}\underline{\xi_0}\right) \\ + \sum_{i\in I_e} d_i \left(-n\log\frac{r_1}{r_0} + h_{\mathrm{ex}} c_0 {r_1}^2\right) \\ + \sum_{i\in I_{\mathcal{A}}} d_i \left(-n\log\frac{r_i}{r_0} + h_{\mathrm{ex}} c_0 {r_i}^2\right).\end{aligned} \tag{9.48}$$

Here we have also used the fact that since we assumed $D^2\xi_0(p)$ is positive definite, there exists $c_0 > 0$ such that if $|b_i - p| < \delta$ and δ is small enough, then $\xi_0(b_i) - \underline{\xi_0} \geq c_0|b_i - p|^2$.

It remains to bound from below each of the above four sums, that we call respectively S_0, S_-, S_e and $S_{\mathcal{A}}$. We leave S_0 unchanged since it corresponds to a term we wish to see in the right-hand side of (9.40).

Concerning S_- and S_e, we first note that since $r_1/r_0 = \delta/K\ell$ and from the definition of ℓ we have

$$\frac{n}{h_{\text{ex}}} \log \frac{r_1}{r_0} = \ell^2 \log \frac{\delta}{K\ell},$$

which is smaller than both $|\underline{\xi_0}|/2$ and $c_0\delta^2/2$ if ℓ small enough depending on K, δ, Ω. Assuming this, and factoring h_{ex} in S_- and S_e we get

$$S_- \geq \frac{1}{2} h_{\text{ex}} D_- |\underline{\xi_0}|, \quad S_e \geq \frac{1}{2} h_{\text{ex}} D_e c_0 \delta^2. \tag{9.49}$$

It remains to investigate $S_{\mathcal{A}}$. For this we factor $h_{\text{ex}}\ell^2$ to find

$$S_{\mathcal{A}} \geq h_{\text{ex}}\ell^2 \sum_{i\in I_{\mathcal{A}}} d_i \left(c_0 (r_i/\ell)^2 - \log \frac{r_i}{K\ell} \right),$$

and thus if K is chosen large enough we find

$$S_{\mathcal{A}} \geq h_{\text{ex}}\ell^2 \sum_{i\in I_{\mathcal{A}}} d_i \frac{c_0}{2} \frac{r_i^2}{\ell^2} = n \sum_{I\in I_{\mathcal{A}}} d_i \frac{c_0}{2} \frac{r_i^2}{\ell^2}. \tag{9.50}$$

From (9.48) and the positivity of the right-hand sides of (9.49) and (9.50) we immediately deduce (9.44).

Now if we assume that the difference between the left-hand side and the right-hand side of (9.44) is less than Cn^2, then this means that $\Delta - 2\pi S_0 \leq Cn^2$ and therefore in view of (9.48) that $S_- + S_e + S_{\mathcal{A}} \leq Cn^2$. In this case we deduce from (9.49) that, as claimed,

$$D_- \leq C\frac{n^2}{h_{\text{ex}}}, \quad D_e \leq C\frac{n^2}{h_{\text{ex}}}.$$

To get (9.45) we note that

$$|D(t) - n| \leq D_- + D_e + \sum_{\substack{d_i>0\\ t<r_i<r_1}} d_i \leq C\frac{n^2}{h_{\text{ex}}} + Cn\frac{\ell^2}{t^2},$$

where we have used (9.50) to bound $\sum_{t<r_i<r_1} |d_i|$. Then noting that $\ell^2 = n/h_{\text{ex}}$ proves (9.45) and the lemma. □

Proof of Proposition 9.4. Proposition 9.4 follows straightforwardly from Lemmas 9.3 and 9.4 if we take note that in (9.43), the term $Cn^2\varepsilon^{\alpha/4}\log(r_1/r_0)$ is $o(n^2)$. The only statement which does not follow directly is the last assertion (9.42). It is proved using Lemma 9.3 together with the information from (9.41) that we have on the function $D(t)$.

Indeed since $n^2 - D^2 \le 2n(n-D)$, we deduce from (9.41) that

$$\begin{aligned}\int_{r_0}^{r_1} \frac{n^2 - D^2(t)}{t}\,dt &\le Cn^2 \int_{r_0}^{r_1} \frac{1}{t}\left(\frac{\ell^2}{t^2} + \ell^2\right) dt \\ &\le Cn^2\ell^2\left(\frac{1}{{r_0}^2} + \log\frac{1}{r_0}\right) \\ &= C\frac{n^2}{K^2},\end{aligned}$$

if ℓ is small enough depending on K, where we have used the fact that $-\ell^2\log r_0 = -\ell^2\log(K\ell) \to 0$ as $\ell \to 0$, and therefore can be made smaller than $1/K^2$ by choosing ℓ small enough. Inserting this into (9.43), we find (9.42) and the proposition is proved. □

9.3.3 Compactness and Lower Bounds Results

Notation. *We need to introduce some more notation before we proceed. We define, in addition to* $A' = A - h_{ex}\nabla^\perp h_0$*, and dropping the* ε *subscripts,*

$$j' = (iu, \nabla u - iA'u), \quad h' = \operatorname{curl} A', \quad \hat{j} = f(|u|)j', \tag{9.51}$$

where $f(x) = 1$ *if* $x \le 1$ *and* $f(x) = \frac{1}{x}$ *if* $x \ge 1$*. Otherwise stated,* $\hat{j} = j'$ *if* $|u| \le 1$ *and otherwise* $\hat{j} = \rho(\nabla\varphi - A)$*, where we have written* $u = \rho e^{i\varphi}$*.*

Given $\delta > 0$*, we write*

$$B_\delta = B(p,\delta), \quad \Omega_\delta = \Omega \setminus B_\delta. \tag{9.52}$$

Finally, we denote by G_p *the solution of*

$$\begin{cases} -\Delta G_p + G_p = \delta_p & \text{in } \Omega \\ \qquad\quad G_p = 0 & \text{on } \partial\Omega \end{cases} \tag{9.53}$$

As in the previous sections, $\mathcal{B}'$ *and* $\mathcal{B}$ *denote the small and large balls respectively.*

Proposition 9.5. *Assume that* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *satisfies* $F_\varepsilon(u_\varepsilon, A_\varepsilon) < \varepsilon^{-1/4}$ *and that* $h_{ex} < \varepsilon^{-1/8}$*. In particular* (9.5) *is satisfied with* $\alpha = 3/4$*. We also assume that*

$$G_\varepsilon(u, A) \leq f_\varepsilon(n) + Cn^2, \tag{9.54}$$

$1 \ll n \ll h_{ex}$*, and we make one of the following two assumptions:*

$$h_{ex} \leq C|\log \varepsilon| \quad or \quad n' = n. \tag{9.55}$$

Then the following holds.

A) Using the notation (9.51)*, there exists* j_* *and* h_* *such that up to extraction of a subsequence, as* $\varepsilon \to 0$*,*

$$\frac{1}{n}\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}} \rightharpoonup j_*, \quad \frac{h'}{n} \rightharpoonup h_*, \quad \frac{\mu(u, A')}{2\pi n} \rightharpoonup \delta_p, \tag{9.56}$$

weakly in $L^2_{loc}(\Omega \setminus \{p\})$*, weakly in* $L^2(\Omega)$ *and in the dual of* $C^{0,\gamma}_0(\Omega)$ *for some* $\gamma \in (0,1)$*, respectively. Moreover* $j_* \in L^q(\Omega)$ *for any* $q < 2$ *and*

$$\operatorname{curl} j_* + h_* = 2\pi\delta_p. \tag{9.57}$$

Finally, as $\delta \to 0$*,*

$$\liminf_{\varepsilon\to 0} \frac{F_\varepsilon\left(u, A', \Omega \setminus (B_\delta \cup \mathcal{B})\right)}{n^2} \geq \pi \log \frac{1}{\delta} + \pi S_\Omega(p,p) + \frac{1}{2}\int_{\Omega\setminus B_\delta} |j_* + 2\pi\nabla^\perp G_p|^2 + |h_* - 2\pi G_p|^2 + o_\delta(1). \tag{9.58}$$

B) Defining φ *as in* (9.11)*, denoting by* $\tilde{\mu}$ *the push-forward of the measure* $\mu(u, A')$ *by* φ*, and letting also* $\tilde{j} = \ell(\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}}) \circ \varphi^{-1}$*, where* ℓ *was defined in (9.8); we have*

$$\frac{1}{n}\tilde{j} \rightharpoonup J_* \qquad \frac{\tilde{\mu}}{2\pi n} \rightharpoonup \mu_* \tag{9.59}$$

weakly in $L^2_{loc}(\mathbb{R}^2)$ *and in the dual of* $C^{0,\gamma}_c(\mathbb{R}^2)$ *respectively, for some* $\gamma \in (0,1)$*. Moreover* μ_* *is a probability measure and*

$$\operatorname{curl} J_* = 2\pi\mu_*. \tag{9.60}$$

Finally, as $K \to +\infty$,

$$\liminf_{\varepsilon\to 0} \frac{1}{2n^2} \int\limits_{B(p,K\ell)\setminus\mathcal{B}} |\nabla_{A'} u|^2 \geq \pi \log K - \pi \iint \log|x-y|\, d\mu_*(x) d\mu_*(y) + o_K(1). \quad (9.61)$$

The rest of this section is devoted to the proof of this proposition. The first step in the proof is to compare the lower bounds given by Proposition 9.3 with the upper bound (9.54). In our case, $\alpha = 3/4$ while $h_{\rm ex} < \varepsilon^{-1/8}$. This implies that the terms $C h_{\rm ex} \varepsilon^{\frac{3\alpha}{2}-1}$ and $C {h_{\rm ex}}^2 \varepsilon^\alpha$ in (9.24) are $O(1)$ hence $o(n^2)$. Noting that $r = {h_{\rm ex}}^{-\frac12}$, we may then rewrite (9.24) as

$$G_\varepsilon(u,A) \geq {h_{\rm ex}}^2 J_0 + 2\pi h_{\rm ex} \int \xi_0\, d\nu + F_\varepsilon(u,A') - C(n'-n)\sqrt{h_{\rm ex}} + o(n^2). \quad (9.62)$$

We may also simplify (9.25) by recalling that $r = \ell/\sqrt{n}$ hence $\log(r/n\varepsilon) = \log(\ell/\varepsilon) - \frac32 \log n$. Thus (9.25) yields

$$F_\varepsilon(u,A') \geq \pi n \log\frac{\ell}{\varepsilon} + F_\varepsilon(u,A',\Omega\setminus\mathcal{B}) + \frac{1-r^2}{2}\int\limits_{\mathcal{B}} h'^2 + C(n'-n)|\log\varepsilon| + o(n^2), \quad (9.63)$$

where $C > 0$ does not depend on ε. Also recall hypothesis (9.55): If $n \neq n'$, then $h_{\rm ex} = O(|\log\varepsilon|)$ hence $\sqrt{h_{\rm ex}} \ll |\log\varepsilon|$ and $C(n'-n)|\log\varepsilon| - C(n'-n)\sqrt{h_{\rm ex}} \geq \frac12 C(n'-n)|\log\varepsilon| \geq 0$. Comparing (9.62), (9.63) with (9.54), we then deduce

$$2\pi n h_{\rm ex}\underline{\xi_0} + \pi n^2 \log\frac{1}{\ell} + Cn^2 \geq 2\pi h_{\rm ex}\int \xi_0\, d\nu + F_\varepsilon(u,A',\Omega\setminus\mathcal{B}) + \frac{1-r^2}{2}\int\limits_{\mathcal{B}} h'^2 + C(n'-n)|\log\varepsilon|. \quad (9.64)$$

Since $2\pi h_{\rm ex}\int \xi_0\, d\nu = 2\pi h_{\rm ex}\sum_i d_i \xi_0(b_i) \geq 2\pi n h_{\rm ex}\underline{\xi_0}$, it follows that

$$(n'-n) \leq C\frac{n^2 \log\frac{1}{\ell}}{|\log\varepsilon|} = o(n), \quad (9.65)$$

because $n\frac{\log\frac{1}{\ell}}{|\log\varepsilon|}\le C\frac{n}{h_{\rm ex}}\log\frac{h_{\rm ex}}{n}=o(1)$ in view of the assumption $n\ll h_{\rm ex}$.

Convergence of $\mu(u,A')$ and $\tilde\mu$

First, from Lemma 9.2 there exists points $\{b_i\}_i$ in the large balls such that (9.32) holds. As in Lemma 9.2, we let

$$\nu=\sum_i d_i\delta_{b_i},\quad \nu'=\sum_j d_j'\delta_{a_j'}.$$

In our case, the radius of the small balls is $r'=C\varepsilon^{3/8}$ since (9.5) is satisfied with $\alpha=3/4$. Thus applying Theorem 6.2 yields

$$\|\mu(u,A')-2\pi\nu'\|_{(C_0^{0,\gamma})^*}\le Cr'^{\gamma}F_\varepsilon(u,A)\le\varepsilon^{3\gamma/8}\varepsilon^{-1/4}.$$

In other words, for any test-function $\xi\in C_0^{0,\gamma}(\Omega)$,

$$\left|\int_\Omega\xi\,d\left(\mu(u,A')-2\pi\nu'\right)\right|\le C\varepsilon^{\frac{3\gamma-2}{8}}\|\xi\|_{C^{0,\gamma}(\Omega)}. \tag{9.66}$$

Let us now change scales, and consider $\tilde\mu$ and $\tilde\nu'$, the measures pushed forward under φ. Let ζ be a compactly supported test-function. We have, using (9.66),

$$\left|\int_{\mathbb{R}^2}\zeta\,d(\tilde\mu-2\pi\tilde\nu')\right|=\left|\int_\Omega\zeta\left(\frac{x-p}{\ell}\right)\,d\left(\mu-2\pi\nu'\right)\right|\le C\ell^{-\gamma}\varepsilon^{\frac{3\gamma-2}{8}}\|\zeta\|_{C^{0,\gamma}}$$

hence we deduce, for $\gamma\in(0,1)$ close enough to 1 that

$$\left\|\tilde\mu(u,A')-2\pi\tilde\nu'\right\|_{(C_c^{0,\gamma})^*}\le C\ell^{-\gamma}\varepsilon^{\frac{3\gamma-2}{8}}=o(1). \tag{9.67}$$

That the right-hand side is $o(1)$ follows from the fact $\ell^{-\gamma}\le {h_{\rm ex}}^{\gamma/2}\le\varepsilon^{-\gamma/16}$ hence the right-hand side is bounded above by $\varepsilon^{\frac{5\gamma-4}{16}}$, which is $o(1)$ if $\gamma>4/5$.

Secondly, we claim that

$$\frac{\tilde\nu-\tilde\nu'}{n}\rightharpoonup 0 \tag{9.68}$$

weakly as measures. Indeed, given any continuous compactly supported test function f, we have

$$d_i f(\tilde{b}_i) = \sum_{a'_j \in B_i} d'_j \left(f(\tilde{a}'_j) + f(\tilde{b}_i) - f(\tilde{a}'_j) \right), \tag{9.69}$$

where $\tilde{a}'_j = \varphi(a'_j)$, $\tilde{b}_i = \varphi(b_i)$. Using the fact that f is uniformly continuous and that, if $a'_j \in B_i$, then

$$|\tilde{b}_i - \tilde{a}'_j| = \frac{|b_i - a'_j|}{\ell} \leq \frac{1}{\ell\sqrt{h_{\mathrm{ex}}}} = \frac{1}{\sqrt{n}},$$

we obtain, since n tends to $+\infty$,

$$\sum_i \sum_{a'_j \in B_i} d'_j |f(\tilde{b}_i) - f(\tilde{a}'_j)| = o(n') = o(n),$$

in view of (9.65), which together with (9.69) yields

$$\frac{1}{n}\sum_i d_i f(\tilde{b}_i) = \frac{1}{n}\sum_i d'_i f(\tilde{a}'_i) + o(1),$$

hence the weak convergence of $(\tilde{\nu} - \tilde{\nu}')/n$ to zero.

Third, we prove the narrow convergence of $\{\tilde{\nu}/n\}_\varepsilon$ to a probability measure. We apply Proposition 9.4. The right-hand side of (9.64) is greater than the left-hand side of (9.40) while the upper bound in (9.64) and the lower bound in (9.40) differ by at most $O(n^2)$. It then follows from Proposition 9.4 that (9.41) is satisfied. From the definitions (9.38) and (9.39),

$$\sum_{|b_i - p| > t} |d_i| \leq n - D(t) + 2D^-,$$

hence it follows from (9.41) and the fact that $D^- = o(n)$ that

$$|\tilde{\nu}|(\mathbb{R}^2 \setminus B(0,M)) = \sum_{|b_i - p| > M\ell} |d_i| \leq C\frac{n}{M^2} + o(n).$$

This proves the narrow convergence of $\{\tilde{\nu}/n\}_\varepsilon$. Moreover, since $D^- = o(n)$, the negative part of $\tilde{\nu}/n$ goes to zero, hence the limit μ_* of $\tilde{\nu}/n$ is a probability measure.

To conclude, from (9.68), (9.67) and the above, the measures $\frac{\tilde{\mu}}{2\pi n}$ converge, in $(C_c^{0,\gamma}(\mathbb{R}^2))^*$ to μ_* which is a probability measure, proving the second part of (9.59). The convergence of the original (i.e., before blow-up) measures $\mu(u, A')/(2\pi n)$ as stated in (9.56) follows by blow-down, since $\ell = o(1)$.

Convergence of $\hat{j}$, h' and $\tilde{j}$

Comparing (9.64) and (9.40) again, we find

$$F_\varepsilon(u, A', \Omega \setminus (\mathcal{A} \cup \mathcal{B})) + \left(\frac{1}{4} - \frac{1}{2h_{\mathrm{ex}}}\right) \int_{\mathcal{A}\cup\mathcal{B}} |h'|^2 \leq Cn^2 + \pi n^2 \log \frac{K}{\delta}. \quad (9.70)$$

Since $h_{\mathrm{ex}} \gg 1$, we find that h'/n is bounded in $L^2(\Omega)$ and up to extraction $\frac{h'}{n} \rightharpoonup h_*$ weakly in $L^2(\Omega)$.

We turn to $\hat{j}$ and $\tilde{j}$. We know that $|j'| \leq |u||\nabla_{A'} u|$. Therefore, we have $|\hat{j}| \leq |\nabla_{A'} u|$, and, in view of (9.70),

$$\int_{\Omega\setminus(\mathcal{A}\cup\mathcal{B})} |\hat{j}|^2 \leq \int_{\Omega\setminus(\mathcal{A}\cup\mathcal{B})} |\nabla_{A'} u|^2 \leq 2F_\varepsilon(u, A', \Omega \setminus (\mathcal{A} \cup \mathcal{B})) \leq Cn^2 \left(1 + \log \frac{K}{\delta}\right).$$

Keeping δ and K fixed, and since $\mathcal{A} \subset B_\delta$, this implies that

$$\left\| (1/n)\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}} \right\|_{L^2(\Omega_\delta)} \leq C \left(\log \frac{K}{\delta} + 1\right). \quad (9.71)$$

Using a diagonal argument, this implies the convergence of a subsequence to some j_* weakly in $L^2_{\mathrm{loc}}(\Omega \setminus \{p\})$. That $j_* \in L^q(\Omega)$ for any $q < 2$ follows as in [189] by writing $\Omega \setminus \{p\} = \cup_n U_n$, where U_n is the set of $x \in \Omega$ such that $2^{-n-1} \leq |x - p| \leq 2^{-n}$, and then estimating in each U_n the L^q norm of j_* in terms of the L^2 norm using Hölder's inequality. Using (9.71) this allows us to prove that $\sum_n \|j_*\|^q_{L^q(U_n)}$ converges, hence that $j_* \in L^q(\Omega)$. We leave the details to the reader.

As for $\tilde{j}$, the above also tells us that for any $K \geq K_0$,

$$\int_{B(p,K\ell)\setminus\mathcal{B}} |\hat{j}|^2 = \int_{B(p,K\ell)} |\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}}|^2 \leq Cn^2,$$

where the constant depends on K. But since $\tilde{j} = \ell\left(\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}} \circ \varphi^{-1}\right)$, this is the integral of $|\tilde{j}|^2$ over $B(0,K)$, hence again using a diagonal argument, this implies the convergence of a subsequence to some J_* weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$, which is the first part of (9.59).

Proof of $\operatorname{curl} j_* + h_* = 2\pi\delta_p$**.**

We begin with the following preliminary result:

Proposition 9.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. For any finite collection of disjoint closed balls $\{B_i\}_{i\in I}$ in $\mathbb{R}^2$ there exists $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$, constant on each ball, such that*

$$\forall \gamma \in [0,1], \quad \|\tilde{f} - f\|_{C^{0,\gamma}(\Omega)} \leq C_\Omega r^{1-\gamma}, \tag{9.72}$$

$$\|\nabla(\tilde{f} - f)\|_{L^\infty(\Omega)} \leq C_\Omega, \quad \|\nabla(\tilde{f} - f)\|_{L^1(\Omega)} \leq C_\Omega r, \tag{9.73}$$

for any bounded $\Omega \subset \mathbb{R}^2$, where C_Ω depends on f, q, and Ω only, and r is the sum of the radii of the balls $\{B_i\}_{i\in I}$.

Moreover, if f is constant in $B(x, \sqrt{2}r)$, then $\tilde{f}(x) = f(x)$.

Proof. Let us write the projection A_1 of $\cup_{i\in I} B_i$ on the first coordinate axis as a disjoint union of closed intervals $[\alpha_1, \beta_1] \cup \cdots \cup [\alpha_n, \beta_n]$. The sum of their lengths $\sum_i \beta_i - \alpha_i$ is smaller than $2r$. We define the function $\varphi_1 : \mathbb{R} \to \mathbb{R}$ by $\varphi_1(0) = 0$ and

$$\varphi_1'(x) = \begin{cases} 0 & \text{if } x \in \cup_{i=1}^n [\alpha_i, \beta_i] \\ 1 & \text{otherwise.} \end{cases}$$

Hence φ_1 is constant on each interval $[\alpha_i, \beta_i]$ and approximates the identity in the sense that $|\varphi_1(x) - x| \leq 2r$, while $|{\varphi_1}'(x)| \leq 1$ for any x. Similarly we can define φ_2 by using the projection A_2 of $\cup_i B_i$ on the second coordinate axis, and φ_2 will satisfy the same properties.

We set $\varphi(x, y) = (\varphi_1(x), \varphi_2(y))$. It is clear that $\varphi(x)$ is constant on each B_i and that $|\varphi(x) - x| \leq 2\sqrt{2}r$. Moreover, $D\varphi(x) = \mathrm{Id}$ outside

$A = (A_1 \times \mathbb{R}) \cup (A_2 \times \mathbb{R})$, while $|D\varphi(x)| \leq 1$ in A. Note also that $|A \cap \Omega| \leq C_\Omega r$.

Given a smooth function f, we let $\tilde{f} = f \circ \varphi$. Then using the fact that $|\varphi(x) - x| \leq C\sqrt{2}r$, we easily deduce that $\|\tilde{f} - f\|_{C^0(\Omega)} \leq C_\Omega r$ (we can take C_Ω to be $2\sqrt{2}$ times the Lipschitz norm of f in Ω) and that if f vanishes in $B(x, \sqrt{2}r)$, then $\tilde{f}(x) = 0$. To prove the gradient bound, we write

$$\nabla(\tilde{f}-f) = (D\varphi)^t \nabla f(\varphi) - \nabla f = (D\varphi)^t \left(\nabla f(\varphi) - \nabla f\right) + \left((D\varphi)^t - \mathrm{Id}\right) \nabla f.$$

The first term is bounded above in Ω by $C_\Omega r$, where C_Ω is $2\sqrt{2}$ times the Lipschitz norm of Df in Ω. The second term is bounded in $A \cap \Omega$ and 0 in the complement. Therefore, the sum is bounded in L^∞, and its L^1 norm is bounded by $C_\Omega r + C_\Omega |A \cap \Omega| \leq C_\Omega r$. The $C^{0,\gamma}$ convergence of $\tilde{f} - f$ follows immediately by interpolation between C^0 and $C^{0,1}$. □

We now prove (9.57), i.e., that for any $f \in \mathcal{D}(\Omega)$,

$$-\int_\Omega j_* \cdot \nabla^\perp f + \int_\Omega h_* f = 2\pi f(p).$$

By approximation, we can assume that f is constant in a neighborhood of p and then using (9.56) we are reduced to proving that

$$\lim_{\varepsilon \to 0} \frac{1}{n} \int_\Omega -\mathbf{1}_{\Omega \setminus \mathcal{B}} \hat{j} \cdot \nabla^\perp f + h' f = 2\pi f(p).$$

Since we already know that $(\operatorname{curl} j' + h')/n = \mu(u, A')/n$ converges to $2\pi\delta_p$, the above equality is true if we replace $\mathbf{1}_{\Omega \setminus \mathcal{B}} \hat{j}$ with j'. It remains to show that

$$\lim_{\varepsilon \to 0} \frac{1}{n} \int_\Omega \left(\mathbf{1}_{\Omega \setminus \mathcal{B}} \hat{j} - j'\right) \cdot \nabla^\perp f = 0. \tag{9.74}$$

Let us define $\tilde{f}$ through Proposition 9.6, using the large balls $\cup_i B_i$ as the collection of balls. Note that since the collection of balls depends on ε, so does $\tilde{f}$, even though f does not. We write

$$\left(\mathbf{1}_{\Omega \setminus \mathcal{B}} \hat{j} - j'\right) \cdot \nabla^\perp f = \mathbf{1}_{\Omega \setminus \mathcal{B}} \hat{j} \cdot \nabla^\perp (f - \tilde{f}) + (\mathbf{1}_{\Omega \setminus \mathcal{B}} - 1) \hat{j} \cdot \nabla^\perp \tilde{f} \\ + (\hat{j} - j') \cdot \nabla^\perp \tilde{f} + j' \cdot \nabla^\perp (\tilde{f} - f) \tag{9.75}$$

and prove that the contribution of each of the four terms to the limit (9.74) is null.

Since the total radius of the balls goes to zero as $\varepsilon \to 0$ and since f is constant in a neighborhood of p, the last assertion of Proposition 9.6 ensures that if δ is chosen small enough, then $\tilde{f}$ and f are constant in $B(p,\delta)$ for any ε. Thus, using (9.71) and (9.73) we find

$$\frac{1}{n}\int_\Omega \mathbf{1}_{\Omega\setminus\mathcal{B}}\hat{j}\cdot\nabla^\perp(f-\tilde{f}) = o(1).$$

Also, since $\tilde{f}$ is constant on each of the balls, $(\mathbf{1}_{\Omega\setminus\mathcal{B}} - 1)\hat{j}\cdot\nabla^\perp\tilde{f} = 0$.

Concerning the term $j'\cdot\nabla^\perp(\tilde{f}-f)$, we know from (9.72) that $\tilde{f}-f$ tends to zero in $C_0^{0,\gamma}$ for any $\gamma\in[0,1]$. But we know that $(\operatorname{curl} j')/n$ converges in the dual of $C_0^{0,\gamma}$, for some γ therefore,

$$\frac{1}{n}\int_\Omega j'\cdot\nabla^\perp(\tilde{f}-f) = o(1).$$

Finally, and in view of (9.75) this will conclude the proof of (9.74) and (9.57), we show that

$$\frac{1}{n}\int_\Omega (\hat{j}-j')\cdot\nabla^\perp\tilde{f} = o(1). \tag{9.76}$$

The proof follows arguments used in the proof of Theorem 6.2. We observe that $j'-\hat{j} = \chi(|u|)\hat{j}$ where $\chi(x)=0$ if $x\le 1$ and $\chi(x)=x-1$ if $x\ge 1$. It follows that, where $|u|>1$,

$$|j'-\hat{j}| \le (|u|-1)|\hat{j}| \le (|u|^2-1)|\hat{j}|.$$

Integrating over the set $\{|u|>1\}$ and using the Cauchy–Schwarz inequality together with the inequality $|\hat{j}|\le|\nabla_{A'}u|$ we obtain that

$$\int_\Omega |j'-\hat{j}| \le C\varepsilon F_\varepsilon(u,A')^{\frac{1}{2}} = o(1),$$

hereby proving (9.76).

Proof of $\operatorname{curl} J_* = 2\pi\mu_*$**.**

We already proved that $\frac{1}{n}\tilde{\mu} \rightharpoonup 2\pi\mu_*$ in the dual of $C_c^{0,\gamma}$, for appropriate γ. Let f be a smooth compactly-supported test-function. We approximate as above, f, using Proposition 9.6 by functions $\tilde{f}$ which are constant on each of the rescaled balls $\varphi(B_i)$. Since the total radius of the balls $\{B_i\}_i$ is ${h_{\rm ex}}^{-1/2}$, the total radius of $\{\varphi(B_i)\}_i$ is $n^{-1/2}$ which is $o(1)$ and therefore we have that $\tilde{f} - f$ converges to 0 locally in H^1 and in $C^{0,\gamma}$.

From the $(C_c^{0,\gamma})^*$ convergence of $\frac{1}{n}\tilde{\mu}$ to $2\pi\mu_*$, we have

$$\frac{1}{n}\int \tilde{f}\,\tilde{\mu} \to 2\pi \int f\,d\mu_*. \tag{9.77}$$

But, by definition of $\tilde{\mu}$,

$$\int \tilde{f}\tilde{\mu} = \int_\Omega (\tilde{f}\circ\varphi)\mu(u,A') = \int_\Omega -\nabla^\perp(\tilde{f}\circ\varphi)\cdot j' + (\tilde{f}\circ\varphi)h',$$

where $\varphi(y) = (y-p)/\ell$. Changing variables, we get

$$\int \tilde{f}\tilde{\mu} = \int_{\varphi(\Omega)} -\ell j'\,(p+\ell x)\cdot\nabla^\perp\tilde{f}(x) + \ell^2 \int_{\varphi(\Omega)} h'\,(p+\ell x)\,\tilde{f}(x).$$

Therefore, dividing by n, using (9.77) and the fact that $\tilde{f}$ is constant on the balls,

$$\int_{\varphi(\Omega)} -\frac{\ell}{n}\mathbf{1}_{\Omega\setminus\mathcal{B}}\,(p+\ell x)\,j'\,(p+\ell x)\cdot\nabla^\perp\tilde{f}(x) + \frac{1}{h_{\rm ex}}\int_{\varphi(\Omega)} h'\,(p+\ell x)\,\tilde{f}(x)$$
$$\to 2\pi\int f\,d\mu_*(x).$$

In other words, we have

$$\int_{\varphi(\Omega)} -\frac{1}{n}\tilde{j}\,(x)\cdot\nabla^\perp\tilde{f}(x) + \frac{1}{h_{\rm ex}}\int_{\varphi(\Omega)} h'\,(p+\ell x)\,\tilde{f}(x) \to 2\pi\int f\,d\mu_*(x).$$

The second term in the left-hand side tends to 0 from the bound $\int |h'|^2 \le Cn^2$ and a rescaling. Then, using the strong H^1 convergence of $\tilde{f}$ to f and the weak L^2 convergence of $\tilde{j}/n$, we are led to

$$-\int_{\mathbb{R}^2} J_*\cdot\nabla^\perp f = 2\pi\int f\,d\mu_*(x)$$

which proves (9.60).

Lower bounds on the energy

It remains to prove (9.58) and (9.61).

It follows from the weak L^2 convergence of $\frac{1}{n}\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}}$ and $\frac{1}{n}h'$ that for any $\delta > 0$ small enough

$$\liminf_{\varepsilon\to 0} \frac{1}{n^2}\int_{\Omega_\delta} |\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}}|^2 + |h'|^2 \geq \frac{1}{2}\int_{\Omega_\delta} |j_*|^2 + h_*^2,$$

where we recall Ω_δ is defined by (9.52). Then, using again the inequality $|\nabla_{A'}u| \geq |\hat{j}|$, this yields

$$\liminf_{\varepsilon\to 0} \frac{1}{n^2} F_\varepsilon(u, A', \Omega_\delta \setminus \mathcal{B}) \geq \frac{1}{2}\int_{\Omega_\delta} |j_*|^2 + h_*^2. \tag{9.78}$$

Let us estimate the right-hand side of (9.78). We decompose j_* and h_* by writing

$$j_* = X - 2\pi\nabla^\perp G_p, \quad h_* = f + 2\pi G_p. \tag{9.79}$$

From (9.53) and (9.57) we have $\operatorname{curl} X + f = 0$ in Ω and, since $h' = 0$ on $\partial\Omega$, $f = 0$ on $\partial\Omega$ also, thus if we introduce a Hodge decomposition $X = \nabla\alpha + \nabla^\perp\beta$, where $\alpha \in H^1(\Omega)$ and $\beta \in H_0^1(\Omega)$, the divergence-free part β satisfies $\Delta\beta + f = 0$ in Ω. We recall that G_p is in L^q for every $q < +\infty$ and ∇G_p is in L^p for every $p < 2$. Therefore, $f \in L^2$ and by elliptic regularity $\beta \in W^{2,2}$. Then by Sobolev embedding, $\beta \in W^{1,q}$ for any $1 \leq q < +\infty$.

We decompose the right-hand side of (9.78) according to (9.79) to find

$$\frac{1}{2}\int_{\Omega_\delta} |j_*|^2 + h_*^2 = \frac{1}{2}\int_{\Omega_\delta} (|X|^2 + f^2 + |2\pi\nabla^\perp G_p|^2 + |2\pi G_p|^2) + 2\pi\int_{\Omega_\delta} (-X \cdot \nabla^\perp G_p + f G_p). \tag{9.80}$$

The first integral will give us the desired lower bound, but let us first check that the cross terms tend to zero with δ. For this we use the Hodge decomposition of X. From the above considerations, both $\nabla^\perp\beta \cdot \nabla^\perp G_p$

and fG_p are integrable in Ω, therefore

$$\int_{\Omega_\delta} (-\nabla^\perp \beta \cdot \nabla^\perp G_p + fG_p) = \int_{\Omega} (-\nabla^\perp \beta \cdot \nabla^\perp G_p + fG_p) + o_\delta(1) = o_\delta(1), \quad (9.81)$$

since $\Delta\beta + f = 0$ in Ω and $G_p = 0$ on $\partial\Omega$. On the other hand, from (7.18) we have $2\pi G_p(x) = -\log|x-p| + S_\Omega(p,x)$, where $x \mapsto S_\Omega(p,x)$ is C^1 in Ω and up to the boundary. It follows, using Proposition 9.6 for instance, that we may write $G_p = g_0 + g_1$, where g_0 is constant on $\partial B(p,\delta) \cup \partial\Omega$ and $\|\nabla g_1\|_{L^2(\Omega_\delta)} = o_\delta(1)$. Then we have

$$\int_{\Omega_\delta} \nabla\alpha \cdot \nabla^\perp G_p = \int_{\Omega_\delta} \nabla\alpha \cdot \nabla^\perp g_0 + \int_{\Omega_\delta} \nabla\alpha \cdot \nabla^\perp g_1 = 0 + o_\delta(1). \quad (9.82)$$

Summing (9.81) and (9.82) we find that the right-hand side of (9.80) is equal to

$$\frac{1}{2}\int_{\Omega_\delta} \left(|j_* + 2\pi\nabla^\perp G_p|^2 + |h_* - 2\pi G_p|^2 + |2\pi\nabla^\perp G_p|^2 + |2\pi G_p|^2\right) + o_\delta(1).$$

To conclude that (9.58) holds, it remains to note that

$$\frac{1}{2}\int_{\Omega_\delta} |2\pi\nabla G_p|^2 + |2\pi G_p|^2 = -\pi\log\delta + \pi S_\Omega(p,p) + o_\delta(1),$$

which is a direct computation, using $2\pi G_p(x) = -\log|x-p| + S_\Omega(x,p)$.

The proof of (9.61) follows similar lines. Using $|\nabla_{A'}u| \geq |\hat{j}|$, we find, for $K \geq K_0$,

$$\int_{B(p,K\ell)\setminus\mathcal{B}} |\nabla_{A'}u|^2 \geq \int_{B(p,K\ell)} \left|\hat{j}\mathbf{1}_{\Omega\setminus\mathcal{B}}\right|^2.$$

Rescaling, we find, writing B_K for $B(0,K)$,

$$\liminf_{\varepsilon\to 0} \frac{1}{2n^2} \int_{B(p,K\ell)\setminus\mathcal{B}} |\nabla_{A'}u|^2 \geq \liminf_{\varepsilon\to 0} \frac{1}{2n^2} \int_{B_K} \left|\tilde{j}\right|^2 \geq \int_{B_K} \frac{|J_*|^2}{2}, \quad (9.83)$$

by weak L^2_{loc} convergence of $(1/n)\tilde{j}$ to J_*.

Let us estimate the right-hand side of (9.83). Using complex coordinates in the plane, we introduce

$$U(x) = \int_{B_K} \log \left| \frac{K(x-y)}{K^2 - x\bar{y}} \right| \, d\mu_*(y).$$

Then, as is well known, we have $\Delta U = 2\pi\mu_*$ in B_K and $U = 0$ on ∂B_K. Again let us write a Hodge decomposition $J_* = Y + \nabla^\perp U$. Since $\operatorname{curl} \nabla^\perp U = \Delta U = 2\pi\mu_* = \operatorname{curl} J_*$ we have $\operatorname{curl} Y = 0$. We compute

$$\int_{B_K} |J_*|^2 = \int_{B_K} |Y|^2 + 2Y \cdot \nabla^\perp U + |\nabla^\perp U|^2 \geq \int_{B_K} |\nabla^\perp U|^2, \tag{9.84}$$

where the cross-term has vanished through integration by parts. Now

$$\int_{B_K} |\nabla^\perp U|^2 = -\int_{B_K} U \Delta U = -2\pi \iint_{B_K \times B_K} \log \left| \frac{K(x-y)}{K^2 - x\bar{y}} \right| \, d\mu_*(y) \, d\mu_*(x),$$

and the integrand of the double integral may be rewritten as $-2\pi(\log|x-y| - \log K - \log|1 - x\bar{y}/K^2|)$. Changing variables $v = x/K$, $w = y/K$, we have

$$\iint_{B_K \times B_K} \log|1 - x\bar{y}/K^2| \, d\mu_*(x) \, d\mu_*(y) = \iint_{B_1 \times B_1} \log|1 - v\bar{w}| \, d\mu_K(v) \, d\mu_K(w),$$

where μ_K is the push-forward of μ_* under the mapping $x \mapsto x/K$. In particular $\lim_{K \to +\infty} \mu_K = \delta_0$. We deduce that the above integral is $o_K(1)$ and then, that

$$\int_{B_K} |\nabla^\perp U|^2 = \iint_{B_K \times B_K} -2\pi \left(\log|x-y| - \log K\right) d\mu_*(y) d\mu_*(x) + o_K(1).$$

We deduce, recalling that μ_* is a probability measure,

$$\frac{1}{2} \int_{B_K} |\nabla^\perp U|^2 = -\pi \iint \log|x-y| \, d\mu_*(y) d\mu_*(x) + \pi \log K + o_K(1).$$

Combining this with (9.84) and inserting into (9.83) proves (9.61).

9.3.4 Completing the Proof of Theorem 9.1

Item 1 in Theorem 9.1 follows from Proposition 9.5. We prove item 2.

As in Proposition 9.5, and since the hypotheses of Theorem 9.1 are identical, we again have that (9.62), (9.63) hold. We split the term

$$F_\varepsilon(u, A', \Omega \setminus \mathcal{B}) + \frac{1-r^2}{2} \int_{\mathcal{B}} h'^2$$

in (9.63) by writing

$$\Omega \setminus \mathcal{B} = (\Omega \setminus (B_\delta \cup \mathcal{B})) \cup (\mathcal{A} \setminus \mathcal{B}) \cup (B(0, K\ell) \setminus \mathcal{B}) .$$

Then we may add up the lower bounds (9.40), (9.58) and (9.61) to obtain

$$\begin{aligned} f_\varepsilon(n) + Cn^2 \geq G_\varepsilon(u, A) \geq f_\varepsilon(n) - \pi n^2 \iint \log|x-y| \, d\mu_*(x) \, d\mu_*(y) \\ + 2\pi h_{\text{ex}} \sum_{\substack{|b_i - p| < K\ell \\ d_i > 0}} d_i(\xi_0(b_i) - \underline{\xi_0}) - Cn^2\delta^2 + o(n^2) + o_\delta(1) + o_K(1). \end{aligned} \tag{9.85}$$

As a byproduct of the fact that the upper and lower bounds match up to $O(n^2)$, we obtain that the left- and right-hand side in (9.40) also match up to $O(n^2)$ and therefore that (9.41)–(9.42) hold.

To prove (9.14) it then suffices to add up (9.42), (9.58) and (9.61) and to insert the result into (9.63). Letting K tend to $+\infty$ and δ tend to 0 yields the result.

To prove (9.13) and finish the proof of Theorem 9.1 it remains to show that

$$\liminf_{\varepsilon \to 0} \frac{2\pi h_{\text{ex}}}{n^2} \sum_{\substack{|b_i - p| < K\ell \\ d_i > 0}} d_i(\xi_0(b_i) - \underline{\xi_0}) \geq \pi \int_{B(0,K)} Q(x) \, d\mu_*(x) + o_K(1). \tag{9.86}$$

Indeed inserting (9.86) into (9.85) and letting $\delta \to 0$ and $K \to +\infty$ proves (9.13).

To prove (9.86), we rescale, letting $\tilde{b}_i = \varphi(b_i)$ as before, or equivalently $b_i = p + \ell \tilde{b}_i$. Then, letting $\tilde{\nu} = \frac{1}{n} \sum_i d_i \delta_{\tilde{b}_i}$ and recalling that

$\ell^2 = n/h_{\text{ex}}$ we have

$$\frac{2\pi h_{\text{ex}}}{n^2} \sum_{\substack{|b_i - p| < K\ell \\ d_i > 0}} d_i(\xi_0(b_i) - \underline{\xi_0}) \geq 2\pi \int_{B(0,K)} \frac{1}{\ell^2} \left(\xi_0(p + \ell x) - \xi_0(p)\right) \, d\tilde{\nu}(x).$$

Since $\ell^{-2}\left(\xi_0(p + \ell x) - \xi_0(p)\right)$ converges locally uniformly to $\frac{1}{2}Q(x)$, where $Q = \langle D^2\xi_0(p)x, x\rangle$ and since $\tilde{\nu}$ converges narrowly to μ_* (as seen in the proof of Proposition 9.5), the right-hand side converges as $\varepsilon \to 0$ to

$$\pi \int_{B(0,K)} Q(x) \, d\mu_*(x)$$

and (9.86) follows.

9.4 Minimization with Respect to n

Theorem 9.1 and Proposition 9.1 already tell us what the limiting normalized vorticity measure is for minimizers of the Ginzburg–Landau functional, and even what the blow-up limit is. But we can also determine the normalizing factor n, i.e., the actual number of vortices. We begin by defining the function which is the leading term as $\varepsilon \to 0$ of the minimal energy of a configuration with n vortices.

By analogy with (9.9), we write

$$\begin{aligned} g_\varepsilon(n) &= {h_{\text{ex}}}^2 J_0 + \pi n |\log \varepsilon| - 2\pi n h_{\text{ex}} |\underline{\xi_0}| + \pi(n^2 - n) \log \frac{1}{\ell} \\ &\quad + \pi n^2 S_\Omega(p,p) + n^2 I_0 \\ &= f_\varepsilon(n) + n^2 I_0 \end{aligned} \tag{9.87}$$

where we recall that p is the unique minimum of ξ_0 in Ω and where ξ_0, $\underline{\xi_0}$ and S_Ω are defined respectively in (7.2), (7.4) and (7.18). Finally I_0 is defined by (9.18), J_0 in (7.3) and $\ell = \sqrt{n/h_{\text{ex}}}$. We also let $g_\varepsilon(0) = {h_{\text{ex}}}^2 J_0$.

Theorem 9.1 and Proposition 9.1 described the minimization of the energy when n and h_{ex} are fixed. They imply (see Corollary 9.1) that the minimal energy is precisely $g_\varepsilon(n)$ plus lower order terms under the hypothesis that $1 \ll n \ll h_{\text{ex}} \leq C|\log \varepsilon|$. Our present problem is to minimize for given h_{ex}, but not n. It seems then that it suffices to minimize $g_\varepsilon(n)$ with respect to n to find the optimal number of vortices, but this is rigorous only if we are able to derive the a priori estimate $1 \ll n \ll h_{\text{ex}}$, and if we minimize under this constraint. Indeed $g_\varepsilon(n)$ tends to $-\infty$ as $n \to +\infty$.

We begin with some facts about the minimization of g_ε. We recall from (7.16) the definition $H_{c_1}^0 = \frac{1}{2|\underline{\xi_0}|}|\log\varepsilon|$.

Lemma 9.5. *There exist constants* $\alpha, \varepsilon_0 > 0$ *and for each* $0 < \varepsilon < \varepsilon_0$ *an increasing sequence* $\{H_n\}_n$ *defined for integers* $0 \le n \le \alpha|\log\varepsilon|$*, such that if* $h_{ex} > H_{c_1}^0/2$*, then* n *minimizes* g_ε *over the integers in the interval* $[0, \alpha|\log\varepsilon|]$ *if and only if*

$$h_{ex} \in [H_n, H_{n+1}].$$

Moreover if n *is a function of* ε *satisfying* $1 \ll n \ll |\log\varepsilon|$*, then the following asymptotic expansion holds as* $\varepsilon \to 0$

$$H_n \sim H_{c_1}^0 + \frac{n-1}{2|\underline{\xi_0}|}\log\frac{|\log\varepsilon|}{n}, \tag{9.88}$$

and if $h_{ex} \in [H_n, H_{n+1}]$*,*

$$g_\varepsilon(n) \sim {h_{ex}}^2 J_0 - \pi n^2 \log\frac{1}{\ell} \quad \textit{as } \varepsilon \to 0. \tag{9.89}$$

We can then characterize the behavior of minimizers in this regime.

Theorem 9.2. (Behavior of minimizers in the intermediate regime). *Assume* h_{ex} *is such that*

$$\log|\log\varepsilon| \ll h_{ex}(\varepsilon) - H_{c_1}^0 \ll |\log\varepsilon|,$$

let N_ε *be a corresponding minimizer of* $g_\varepsilon(n)$ *over* $[0, \alpha|\log\varepsilon|]$ *and let* $(u_\varepsilon, A_\varepsilon)$ *minimize* G_ε*. Then for any* $\gamma \in (0,1)$

$$\frac{\tilde\mu(u_\varepsilon, A_\varepsilon)}{2\pi N_\varepsilon} \rightharpoonup \mu_0 \quad \textit{in} \left(C_c^{0,\gamma}(\mathbb{R}^2)\right)^*, \tag{9.90}$$

where μ_0 *is the unique minimizer of* I *and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = g_\varepsilon(N_\varepsilon) + o(N_\varepsilon^2). \tag{9.91}$$

Proof of Lemma 9.5. We let, for any integer $n > 0$,

$$\Delta_n = g_\varepsilon(n) - g_\varepsilon(n-1),$$

and we see Δ_n as a function of $h_{\rm ex}$, ε being fixed.

First, the function Δ_1 is decreasing on $\mathbb{R}_+$ and if $n > 1$, then Δ_n is first increasing and then decreasing. It is not difficult to check that if $h_{\rm ex} = H_{c_1}^0/2$, $n \le \alpha|\log\varepsilon|$ and α, ε are small enough, then Δ_n is strictly positive. Since Δ_n tends to $-\infty$ as $h_{\rm ex} \to +\infty$, we may then define H_n to be the only value of $h_{\rm ex}$ in the interval $[H_{c_1}^0/2, +\infty[$ for which Δ_n vanishes. If $h_{\rm ex} > H_{c_1}^0/2$ and n, ε are as above, then

$$(\Delta_n < 0) \Leftrightarrow (h_{\rm ex} > H_n), \quad (\Delta_n > 0) \Leftrightarrow (h_{\rm ex} < H_n).$$

Second it is easy to check, taking a smaller α if necessary, assuming $h_{\rm ex} > H_{c_1}^0/2$ and $n \le \alpha|\log\varepsilon|$, that for any $\varepsilon > 0$ we have $\Delta_{n+1} - \Delta_n > 0$. In particular the sequence $\{H_n\}_n$ is increasing.

It follows that if we assume $h_{\rm ex} \in [H_n, H_{n+1}]$, then

$$\Delta_1 < \cdots < \Delta_n \le 0 \le \Delta_{n+1} < \cdots < \Delta_k,$$

if $k \le \alpha|\log\varepsilon|$. Recalling that $\Delta_n = g_\varepsilon(n) - g_\varepsilon(n-1)$, this means that for any integer $m \in [0, \alpha|\log\varepsilon|]$ we have $g_\varepsilon(m) \ge g_\varepsilon(n)$. Conversely, if n minimizes g_ε in the interval $[0, \alpha|\log\varepsilon|]$, then $\Delta_n \le 0 \le \Delta_{n+1}$ therefore assuming $h_{\rm ex} > H_{c_1}^0/2$ we must have $h_{\rm ex} \in [H_n, H_{n+1}]$.

To obtain the asymptotic expansion of H_n we write down $\Delta_n(H_n) = 0$ and we get

$$\pi|\log\varepsilon| - 2\pi H_n|\underline{\xi_0}| + \pi(n-1)\log\frac{H_n}{n-1} + \pi\frac{n(n-1)}{2}\log\frac{n-1}{n} \\ + (2n-1)(\pi S_\Omega(p,p) + I_0) = 0.$$

Dividing by $2\pi|\underline{\xi_0}|$ and adding $H_n - H_{c_1}^0$ we find that as $n \to +\infty$

$$\frac{n-1}{2|\underline{\xi_0}|}\log\frac{H_n}{n-1} + O(n) = H_n - H_{c_1}^0. \tag{9.92}$$

But we know that $H_n \ge H_1$ and it is straightforward to check that $H_1 \sim H_{c_1}^0$ as $\varepsilon \to 0$. Therefore if we assume $1 \ll n(\varepsilon) \ll |\log\varepsilon|$, then $n = o(H_n)$ and dividing (9.92) by H_n we find that

$$\frac{H_n - H_{c_1}^0}{H_n} = o(1),$$

and then that $H_n \sim H_{c_1}^0$ as $\varepsilon \to 0$. Plugging this and the expression of $H_{c_1}^0$ into (9.92), we get (9.88). Obviously, when $h_{\text{ex}} \in [H_n, H_{n+1}]$ in this regime $1 \ll n \ll h_{\text{ex}}$, we may also write

$$h_{\text{ex}} = H_{c_1}^0 + \frac{n}{|\underline{\xi_0}|} \log \frac{1}{\ell} + O\left(\log \frac{1}{\ell}\right) + O(n). \tag{9.93}$$

Plugging this into the expression of $g_\varepsilon(n)$ (9.87), we find

$$\begin{aligned} g_\varepsilon(n) &= {h_{\text{ex}}}^2 J_0 - 2\pi n^2 \log \frac{1}{\ell} + \pi(n^2 - n) \log \frac{1}{\ell} + O\left(n \log \frac{1}{\ell}\right) + O(n^2) \\ &= {h_{\text{ex}}}^2 J_0 - \pi n^2 \log \frac{1}{\ell} + O\left(n \log \frac{1}{\ell}\right) + O(n^2). \end{aligned} \tag{9.94}$$

This proves (9.89). □

Proof of the theorem. In a first step, we prove that for minimizers, the total degree is $\ll h_{\text{ex}}$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of G_ε. From Theorem 7.2 and Proposition 7.2, the hypothesis $h_{\text{ex}} - H_{c_1}^0 = o(|\log \varepsilon|)$ implies that $\lambda = \frac{1}{2|\underline{\xi_0}|}$ and $\frac{\mu(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|}$ tends to 0 in $(C^{0,\gamma})^*$ as $\varepsilon \to 0$. Moreover, comparing lower bounds and upper bounds in (7.58) and (7.59), we find that $\sum_i |d_i| = o(|\log \varepsilon|)$, where the d_i's are the degrees of the balls constructed by Theorem 4.1 of size r, for any r such that $|\log r| \ll |\log \varepsilon|$. We deduce that if we consider vortex balls of radius $r = \frac{1}{\sqrt{h_{\text{ex}}}}$, and denote by n their total degree, we have $n \ll h_{\text{ex}}$.

In a second step, we prove that $n \gg 1$. Proposition 9.3 provides a lower bound for G_ε:

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq {h_{\text{ex}}}^2 J_0 + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(b_i) + \pi n \log \frac{\ell}{\varepsilon} - C(1 + n \log n).$$

Using the fact that $d_i \xi_0(b_i) \geq |d_i| \underline{\xi_0}$, we are led to

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq {h_{\text{ex}}}^2 J_0 - 2\pi n h_{\text{ex}} |\underline{\xi_0}| + \pi n |\log \varepsilon| \\ + O(n \log \frac{1}{\ell}) + O(1) + o(n^2). \end{aligned} \tag{9.95}$$

On the other hand, we may construct a comparison map by applying Proposition 9.1 to any $1 \ll N_\varepsilon \ll h_{\text{ex}}$ and to μ_0 the minimizer of I, and find

$$\inf G_\varepsilon \leq f_\varepsilon(N_\varepsilon) + N_\varepsilon^2 I_0 + o(N_\varepsilon^2),$$

and since $(u_\varepsilon, A_\varepsilon)$ is a minimizer of G_ε and $f_\varepsilon(n) + n^2 I_0 = g_\varepsilon(n)$ (see (9.87)) we deduce that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le g_\varepsilon(N_\varepsilon) + o(N_\varepsilon^2), \tag{9.96}$$

where N_ε is chosen to be the minimizer of g_ε over $[0, \alpha|\log\varepsilon|]$, hence $N_\varepsilon \gg 1$ in our regime. Using (9.94), and comparing to (9.95), we find

$$\begin{aligned}&-2\pi n h_{\text{ex}}|\underline{\xi_0}| + \pi n|\log\varepsilon| \\ &\qquad \le -\pi N_\varepsilon^2 \log\frac{1}{L} + O\left(N_\varepsilon \log\frac{1}{L} + N_\varepsilon^2 + n\log\frac{1}{\ell}\right) + o(n^2),\end{aligned}$$

where we have let $L = \sqrt{N_\varepsilon/h_{\text{ex}}}$. Inserting the expansion (9.93), and replacing L by its value we find

$$\begin{aligned}&-\pi n\left(N_\varepsilon \log\frac{h_{\text{ex}}}{N_\varepsilon} + O\left(\log\frac{h_{\text{ex}}}{N_\varepsilon}\right)\right) \\ &\qquad \le -\frac{\pi}{2}N_\varepsilon^2 \log\frac{h_{\text{ex}}}{N_\varepsilon} + O\left(N_\varepsilon\log\frac{h_{\text{ex}}}{N_\varepsilon} + n\log\frac{h_{\text{ex}}}{n} + N_\varepsilon^2 + n^2\right).\end{aligned}$$

Dividing by $N_\varepsilon \log\frac{h_{\text{ex}}}{N_\varepsilon}$ which is $\gg N_\varepsilon$, we are led to

$$n \ge \frac{N_\varepsilon}{2} + O\left(\frac{n}{N_\varepsilon} + 1 + \frac{n\log\frac{h_{\text{ex}}}{n}}{N_\varepsilon\log\frac{h_{\text{ex}}}{N_\varepsilon}}\right) + o(N_\varepsilon + \frac{n^2}{N_\varepsilon}).$$

Writing $\log\frac{h_{\text{ex}}}{n} = \log\frac{h_{\text{ex}}}{N_\varepsilon} + \log\frac{N_\varepsilon}{n}$ we find

$$n \ge \frac{N_\varepsilon}{2} + O\left(1 + \frac{n}{N_\varepsilon}\right) + o\left(\frac{n}{N_\varepsilon}\log\frac{N_\varepsilon}{n} + N_\varepsilon + \frac{n^2}{N_\varepsilon}\right).$$

We can deduce from this relation that $\frac{n}{N_\varepsilon}$ remains bounded below by a positive constant as $\varepsilon \to 0$, hence $n \gg 1$.

Once this is known, we deduce from Theorem 9.1 and Corollary 9.1 the improved estimate

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge g_\varepsilon(n) + o(n^2).$$

Comparing with (9.96), we deduce $g_\varepsilon(N_\varepsilon) \le g_\varepsilon(n) \le g_\varepsilon(N_\varepsilon) + o(N_\varepsilon^2) + o(n^2)$. This implies from (9.94) that

$$\begin{aligned}&-\frac{\pi}{2}n^2\log\frac{h_{\text{ex}}}{n} + O\left(n\log\frac{h_{\text{ex}}}{n}\right) \\ &\qquad = -\frac{\pi}{2}N_\varepsilon^2\log\frac{h_{\text{ex}}}{N_\varepsilon} + O\left(N_\varepsilon\log\frac{h_{\text{ex}}}{N_\varepsilon}\right) + O(n^2 + N_\varepsilon^2).\end{aligned}$$

Writing once more $\log \frac{h_{\rm ex}}{n} = \log \frac{h_{\rm ex}}{N_\varepsilon} + \log \frac{N_\varepsilon}{n}$ and dividing by $\log \frac{h_{\rm ex}}{N_\varepsilon} \gg 1$, we find

$$n^2 - N_\varepsilon^2 = O\left(N_\varepsilon + n\right) + o\left(n \log \frac{N_\varepsilon}{n} + n^2 + N_\varepsilon^2\right).$$

We can finally obtain from this relation that $\frac{n}{N_\varepsilon} \to 1$ as $\varepsilon \to 0$.

From Theorem 9.1 and Corollary 9.1, we deduce (9.90), at least for some $\gamma \in (0,1)$. Also, $G_\varepsilon(u_\varepsilon, A_\varepsilon) = g_\varepsilon(n) + o(n^2) = g_\varepsilon(N_\varepsilon) + o(n^2)$ and (9.91) is proved.

The fact that (9.90) is true for *any* $\gamma \in (0,1)$ follows from Theorem 6.2. Indeed in this regime we have $F_\varepsilon(u_\varepsilon, A_\varepsilon) \le Cn|\log\varepsilon| \le |\log\varepsilon|^2$ and thus applying Theorem 6.2 with $r = \sqrt{\varepsilon}$ we find for any $\gamma \in (0,1)$ that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu\|_{(C_0^{0,\gamma})^*} \le C\varepsilon^{\gamma/2}|\log\varepsilon|^2,$$

where $\nu = 2\pi \sum_i d_i' \delta_{a_i'}$, and (a_i', d_i') are the centers and degrees of the vortex balls of total radius $\sqrt{\varepsilon}$. Letting $n' = \sum_i |d_i'|$ we have moreover, $F_\varepsilon(u, A) \ge Cn'|\log\varepsilon|$ and therefore $n' \le Cn$. Thus $\{\nu/n\}_\varepsilon$ is bounded as Radon measures. If we now rescale and take the push-forwards of μ and ν by $x \mapsto (x-p)/\ell$, we find that $\|\tilde\mu(u_\varepsilon, A_\varepsilon) - \tilde\nu\|_{(C_c^{0,\gamma})^*}$ still goes to zero as $\varepsilon \to 0$ while, of course, $\{\tilde\nu/n\}_\varepsilon$ remains bounded as Radon measures. Hence $\tilde\mu(u, A)/n$ does converge in the dual of $C_c^{0,\gamma}$ as claimed. □

Bibliographic notes on Chapter 9: The energy-splitting result was first observed by Bethuel–Rivière in [51]. The calculation of the first critical field and of the fields H_n was first done in [181]. The other results of the chapter concerning the Γ-convergence in the intermediate regime are new.

Chapter 10

The Case of a Bounded Number of Vortices

In this chapter, we prove upper bound and lower bound estimates for configurations with a number of vortices *bounded* as $\varepsilon \to 0$ which reduces to considering a number of vortices *independent of* ε. These estimates will be useful in the next chapter. The fact that the number of vortices is bounded independently of ε allows us to obtain much more precise information with specific techniques: the upper and lower bounds will match up to an error which is $o(1)$ as $\varepsilon \to 0$.

10.1 Upper Bound

In all that follows, by "bounded away from the boundary" we mean "at a distance from the boundary bounded below by some positive constant".

Proposition 10.1. (Upper bound for a bounded number of vortices). *Assume $n \in \mathbb{N}$ and, for every $\varepsilon > 0$, let $\{a_i^\varepsilon\}_{1\le i\le n}$ be points in Ω bounded away from the boundary and such that $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$ if $i \neq j$. Assume also that $h_{ex}(\varepsilon) \ll \frac{1}{\varepsilon}$. Then for any choice of degrees $d_i \in \{+1, -1\}$ there exists a family of configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ such that, as $\varepsilon \to 0$,*

$$\mu(u_\varepsilon, A_\varepsilon) - 2\pi \sum_{i=1}^{n} d_i \delta_{a_i} \rightharpoonup 0 \quad in\ (C_0^{0,\beta}(\Omega))^*,\ \forall \beta > 0,$$

and

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) = \pi n|\log\varepsilon| - \pi\sum_{i\neq j} d_i d_j \log|a_i^\varepsilon - a_j^\varepsilon| + \pi\sum_{i,j} d_i d_j S_\Omega(a_i^\varepsilon, a_j^\varepsilon) + n\gamma + o(1), \quad (10.1)$$

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = {h_{ex}}^2 J_0 + \pi n|\log\varepsilon| + 2\pi h_{ex}\sum_i d_i \xi_0(a_i^\varepsilon) - \pi\sum_{i\neq j} d_i d_j \log|a_i^\varepsilon - a_j^\varepsilon| + \pi\sum_{i,j} d_i d_j S_\Omega(a_i^\varepsilon, a_j^\varepsilon) + n\gamma + o(1), \quad (10.2)$$

where γ was introduced in (3.15).

The proof of this proposition uses a construction which is very close to the construction of Proposition 7.3. It differs mainly in the way we define $|u|$, and in the precision with which we estimate the energy of the test-configuration.

The Test-Configuration

Let Φ_ε be the solution of

$$\begin{cases} -\Delta\Phi_\varepsilon + \Phi_\varepsilon = 2\pi\sum_i d_i\delta_{a_i^\varepsilon} & \text{in } \Omega \\ \Phi_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (10.3)$$

Dropping the subscript ε, we define A' to be such that $\operatorname{curl} A' = \Phi$. Then

$$\operatorname{curl}(A' - \nabla^\perp\Phi) = 2\pi\sum_i d_i\delta_{a_i}$$

and therefore, denoting by Θ the phase of

$$\prod_{i=1}^n \frac{(z-a_i)^{d_i}}{|z-a_i|^{d_i}},$$

the curl of $A' - \nabla^\perp\Phi - \nabla\Theta$ vanishes in Ω, in the sense of distributions and thus is the gradient of some function g. It follows that, letting $\varphi = \Theta + g$, the function φ is well defined modulo 2π in $\Omega\setminus\{a_1,\dots,a_n\}$ and satisfies

$$\nabla\varphi = A' - \nabla^\perp\Phi. \quad (10.4)$$

Fixing $R > 1$, we define the test configuration $(u_{\varepsilon,R}, A_{\varepsilon,R})$ that we denote in shorthand (u, A) as follows. We let $A = A' + h_{\text{ex}}\nabla^\perp\xi_0$ and we let $u(x) = e^{i\varphi(x)}$ in $\Omega \setminus \cup_i B(a_i, R\varepsilon)$ and for $x \in B(a_i, R\varepsilon)$,

$$u(x) = \frac{1}{f(R)} f\left(\frac{|x-a_i|}{\varepsilon}\right) e^{i\varphi(x)}, \tag{10.5}$$

where f is the modulus of the radial degree-one vortex u_0 introduced in Proposition 3.11. Note that from (3.15) and since $u_0(r,\theta) = f(r)e^{i\theta}$ we have, as $R \to +\infty$,

$$\frac{1}{2}\int_0^R \left(|f'|^2 + \frac{f^2}{r^2} + \frac{(1-f^2)^2}{2}\right) 2\pi r\, dr = \pi \log R + \gamma + o(1). \tag{10.6}$$

Asymptotics for Φ_ε

Let $\{a_i^\varepsilon\}$ satisfy the hypotheses of Proposition 10.1 and Φ_ε be defined by (10.3). Dropping the subscript ε where convenient, we define for $1 \le i \le n$

$$\Phi_i(y) = \Phi_\varepsilon(a_i + \varepsilon y).$$

Then we claim that for any $1 \le i \le n$,

$$\Phi_i(y) = -d_i \log|\varepsilon y| - \sum_{j\neq i} d_j \log|a_i - a_j| + \sum_j d_j S_\Omega(a_i, a_j) + \Delta_{i,\varepsilon}(y), \tag{10.7}$$

where $\{\Delta_{i,\varepsilon}\}_\varepsilon$ converges to zero in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Note that the sums in (10.7) do not depend on y but do depend on ε through the points a_i.

The proof is straightforward. From the definition of S_Ω (7.18), the function Φ_ε can be written more explicitly as

$$\Phi_\varepsilon(x) = -\sum_{i=1}^n d_i \log|x - a_i| + \sum_{i=1}^n d_i S_\Omega(x, a_i) \tag{10.8}$$

and therefore

$$\Delta_{i,\varepsilon}(y) = \sum_{j\neq i} -d_j \log\frac{|\varepsilon y + a_i - a_j|}{|a_i - a_j|} + \sum_j d_j \left(S_\Omega(a_j, a_i + \varepsilon y) - S_\Omega(a_j, a_i)\right).$$

The first sum converges to zero in $C^1_{\text{loc}}(\mathbb{R}^2)$ because we assumed $|a_i - a_j| \gg \varepsilon$. For the second sum, we use the fact that S_Ω is C^1 in $\Omega \times \Omega$ and that the points $a_1, \ldots, a_n$ are bounded away from the boundary. The claim is proved.

Energy Inside the Balls

From (10.4) we have

$$|\nabla_{A'}u|^2 = |u|^2|\nabla\varphi - A'|^2 + |\nabla|u||^2 = |u|^2|\nabla\Phi|^2 + |\nabla|u||^2$$

while $\operatorname{curl} A' = \Phi$. Using (10.5), it follows, by letting $B_i = B(a_i, R\varepsilon)$, and $r = |x-a_i|/\varepsilon$, that

$$F_\varepsilon(u, A', B_i) = \frac{1}{2}\int_{B_i} \frac{f'(r)^2}{\varepsilon^2 f(R)^2} + \frac{f(r)^2}{f(R)^2}|\nabla\Phi(x)|^2 + |\Phi(x)|^2 + \frac{1}{2\varepsilon^2}\left(1 - \frac{f(r)^2}{f(R)^2}\right)^2 dx.$$

Using the change of variables $y = (x-a_i)/\varepsilon$, we have $r = |y|$ and the above becomes

$$F_\varepsilon(u, A', B_i) = \frac{1}{2}\int_{B(0,R)} \frac{f'(r)^2}{f(R)^2} + \frac{f(r)^2}{f(R)^2}|\nabla\Phi_i(y)|^2 + \varepsilon^2|\Phi_i(y)|^2 + \frac{1}{2}\left(1 - \frac{f(r)^2}{f(R)^2}\right)^2 dy.$$

From (10.7) we deduce that $|\nabla\Phi_i(y)|^2 - 1/r^2$ and $\varepsilon^2|\Phi_i(y)|^2$ both converge to zero uniformly in $B(0,R)$, using the fact that $d_i = \pm 1$. Therefore, as $\varepsilon \to 0$,

$$F_\varepsilon(u, A', B_i) = \frac{1}{2}\int_0^R \left(\frac{f'(r)^2}{f(R)^2} + \frac{f(r)^2}{r^2 f(R)^2} + \frac{1}{2}\left(1 - \frac{f(r)^2}{f(R)^2}\right)^2\right) 2\pi r\,dr + o(1).$$

From (10.6) and the fact that $\lim_{+\infty} f = 1$, the integral on the right-hand side is asymptotic as $R \to +\infty$ to $\pi \log R + \gamma$. Therefore

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\left(F_\varepsilon(u, A', \cup_i B_i) - \pi n \log R - n\gamma\right) = 0. \tag{10.9}$$

Energy Outside the Balls

Outside $\cup_i B_i$ and since $|u| = 1$ there, we have $|\nabla_{A'} u|^2 = |\nabla \varphi - A'|^2 = |\nabla \Phi|^2$. Thus, using again the equality $\operatorname{curl} A' = \Phi$, we find

$$F_\varepsilon\left(u, A', \Omega \setminus \cup_i B_i\right) = \frac{1}{2} \int_{\Omega \setminus \cup_i B_i} |\nabla \Phi|^2 + |\Phi|^2 = -\frac{1}{2} \sum_i \int_{\partial B_i} \Phi \frac{\partial \Phi}{\partial \nu}, \tag{10.10}$$

where we have used an integration by parts and the fact that $-\Delta\Phi + \Phi = 0$ in $\Omega \setminus \cup_i B_i$ and $\Phi = 0$ on $\partial\Omega$. Here ν is the unit normal pointing outwards to the ball.

Changing variables as above we have

$$\int_{\partial B_i} \Phi \frac{\partial \Phi}{\partial \nu} = \int_{\partial B(0,R)} \Phi_i \frac{\partial \Phi_i}{\partial \nu}.$$

This integral is easily estimated using the convergence of (10.7). Up to a term converging uniformly to zero on $\partial B(0, R)$ as $\varepsilon \to 0$, the normal derivative of Φ_i is equal to $-d_i/R$ and Φ_i is equal to $-d_i \log|\varepsilon R| - \sum_{j\neq i} d_j \log|a_i - a_j| + \sum_j d_j S_\Omega(a_i, a_j)$. Therefore, as $\varepsilon \to 0$,

$$\int_{\partial B(0,R)} \Phi_i \frac{\partial \Phi_i}{\partial \nu} = 2\pi \log|\varepsilon R| + 2\pi \sum_{j\neq i} d_i d_j \log|a_i - a_j| - 2\pi \sum_j d_i d_j S_\Omega(a_i, a_j) + o(1).$$

Replacing in (10.10) we get

$$\lim_{\varepsilon\to 0} \left(F_\varepsilon\left(u, A', \Omega \setminus \cup_i B_i\right) - \pi n \log\frac{1}{\varepsilon} + \pi n \log R + \pi \sum_{j\neq i} d_i d_j \log|a_i - a_j| - \pi \sum_{i,j} d_i d_j S_\Omega(a_i, a_j) \right) = 0. \tag{10.11}$$

Convergence of $\mu(u, A')$

Using (10.4) and $\operatorname{curl} A' = \Phi$, we have $\mu(u, A') = -\operatorname{curl}(|u|^2\nabla^\perp\Phi) + \Phi$. Since $\Delta\Phi = \Phi$ in $\Omega \setminus \cup_i\{a_i\}$ this becomes $\mu(u, A') = -\nabla|u|^2 \wedge \nabla^\perp\Phi + (1 - |u|^2)\Phi$ and, letting $y = (x - a_i)/\varepsilon$ and $r = |y|$, we find

$$\mu(u, A')\,dx = \left(-\frac{\nabla f(r)^2 \wedge \nabla^\perp\Phi_i}{f(R)^2} + \varepsilon^2\left(1 - |u|^2\right)\Phi_i\right) dy.$$

Using (10.7) the factor in front of dy converges uniformly in $B(0, R)$ as $\varepsilon \to 0$ to $2d_i\dfrac{f(r)f'(r)}{rf(R)^2}$ and we deduce that $\mu(u, A')$ has the *constant* sign d_i in B_i if ε is small enough. Moreover, since $|u| = 1$ on ∂B_i, from Lemma 6.3, the integral of $\mu(u, A')$ over B_i is $2\pi d_i$. It follows that the integral of any continuous function ζ against $\mu(u, A')$ in B_i is equal to $2\pi d_i\zeta(x_i)$, for some $x \in B_i$. Finally, since $|u| = 1$ outside $\cup_i B_i$, we have $\mu(u, A') = 0$ there. We deduce that

$$\left|\int_\Omega \zeta\mu(u, A') - 2\pi\sum_i d_i\zeta(a_i)\right| \le C \max_{|x-y|<R\varepsilon} |\zeta(x) - \zeta(y)|. \tag{10.12}$$

In particular $\mu(u, A') - 2\pi\sum_i d_i\delta_{a_i}$ goes to zero in the dual of $C^0_0(\Omega)$ and its norm in the dual of $C^{0,1}_0(\Omega)$ is smaller than $CR\varepsilon$.

Bounds for G_ε

To evaluate $G_\varepsilon(u, A)$ we invoke Lemma 7.3, which states that

$$G_\varepsilon(u, A) = {h_{\rm ex}}^2 J_0 + h_{\rm ex}\int_\Omega \xi_0\mu(u, A') + F_\varepsilon(u, A') + R_0, \tag{10.13}$$

where

$$R_0 \le C{h_{\rm ex}}^2\int_\Omega(1 - |u|^2).$$

From the definition of $|u|$ we have $|u| = 1$ outside $\cup_i B_i$, while if $x \in B(a_i, R)$ and letting $y = (x-a_i)/\varepsilon$, $r = |y|$, we have $|u(x)| = f(r)/f(R) \ge f(r)$. Then, a change of variables yields

$$\int_{B_i}\left(1 - |u|^2\right)^2 \le \varepsilon^2\int_{B(0,R)}\left(1 - f(r)^2\right)^2 \le 2\pi\varepsilon^2,$$

where we have used (3.14). So with the Cauchy–Schwarz inequality, since n is independent of ε, we find

$$\int_\Omega (1-|u|^2) \le CR\varepsilon \left(\int_{\cup_i B_i} (1-|u|^2)^2 \right)^{\frac{1}{2}} \le CR\varepsilon^2.$$

This, together with (10.12) applied to ξ_0 and (10.13) yields

$$G_\varepsilon(u,A) - F_\varepsilon(u,A') = {h_{\rm ex}}^2 J_0 + 2\pi h_{\rm ex} \sum_i d_i \xi_0(a_i) + O\left(R\varepsilon^2 {h_{\rm ex}}^2 + R\varepsilon h_{\rm ex}\right). \tag{10.14}$$

Diagonal Argument

It follows from (10.9) and (10.11) that we may define radii R_ε tending to $+\infty$ as ε goes to zero and such that, denoting by $(u_\varepsilon, A_\varepsilon)$ the configuration $(u_{\varepsilon,R_\varepsilon}, A_{\varepsilon,R_\varepsilon})$ we have, as $\varepsilon \to 0$,

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le \pi n \log\frac{1}{\varepsilon} - \pi \sum_{j\ne i} d_i d_j \log|a_i - a_j| + \pi \sum_{i,j} d_i d_j S_\Omega(a_i,a_j) + n\gamma + o(1),$$

hereby proving (10.1). We may also assume, since $\varepsilon h_{\rm ex}$ tends to zero and by changing R_ε if necessary, that $R_\varepsilon \varepsilon$, $R_\varepsilon^2 \varepsilon h_{\rm ex}$, $R_\varepsilon \varepsilon h_{\rm ex}$ and thus also $R_\varepsilon \varepsilon^2 {h_{\rm ex}}^2$ tend to 0 with ε. Then in view of (10.14) and (10.1) we have proved (10.2).

To finish, note that from (10.12) and since $\varepsilon R_\varepsilon$ tends to zero, we have $\mu(u_\varepsilon, A'_\varepsilon) - 2\pi \sum_i d_i \delta_{a_i}$ converges to zero in $\left(C_0^0(\Omega)\right)^*$, hence in $(C_0^{0,\beta}(\Omega))^*$ for any $\beta \in (0,1)$. The same is true for $\mu(u_\varepsilon, A_\varepsilon)$. Indeed $\mu(u_\varepsilon, A_\varepsilon) - \mu(u_\varepsilon, A'_\varepsilon) = -h_{\rm ex}\,\mathrm{curl}(|u_\varepsilon|^2 \nabla^\perp \xi_0) + h_{\rm ex} \Delta \xi_0 = h_{\rm ex}\,\mathrm{curl}((1-|u_\varepsilon|^2)\nabla^\perp \xi_0)$, the latter being easily bounded in L^∞ norm by $Ch_{\rm ex}/\varepsilon$, we deduce by integrating over $\cup_i B_i$ that $\mu(u_\varepsilon, A_\varepsilon) - \mu(u_\varepsilon, A'_\varepsilon)$ is bounded in L^1 by $CR_\varepsilon^2 \varepsilon h_{\rm ex}$, which tends to 0 with ε. Thus it converges to 0 in $\left(C_0^0(\Omega)\right)^*$, hence in $(C_0^{0,\beta}(\Omega))^*$. This concludes the proof of Proposition 10.1.

10.2 Lower Bound

Proposition 10.2. (Lower bound for solutions with a bounded number of vortices). *Let* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *be solutions of* (GL) *such that*

$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le C|\log\varepsilon|$ with $h_{ex} \le \varepsilon^{-\beta}$ and $\beta < 1$. If all the (a_i^ε, d_i)'s given by the result of Theorem 5.4 are bounded away from $\partial\Omega$, and are such that $d_i = 1$ for every i, and $\sum_i d_i = n$, then as $\varepsilon \to 0$

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \ge \pi n|\log\varepsilon| + W(a_1^\varepsilon, \dots, a_n^\varepsilon) + n\gamma + o(1), \tag{10.15}$$

where

$$W(a_1, \dots, a_n) = -\pi \sum_{i \ne j} \log|a_i - a_j| + \pi \sum_{i,j} S_\Omega(a_i, a_j)$$

and γ is the constant defined by (3.15).

The hypotheses are such that we may apply Theorem 5.4 and thus find balls $B(a_1^\varepsilon, R_0\varepsilon), \dots, B(a_n^\varepsilon, R_0\varepsilon)$, with n independent of ε such that $|a_i^\varepsilon - a_j^\varepsilon| \gg \varepsilon$ for $i \ne j$, $\mathrm{dist}(a_i^\varepsilon, \partial\Omega) \gg \varepsilon$ and such that $|u| \ge \frac{1}{2}$ in $\Omega \setminus \cup_i B(a_i^\varepsilon, R_0\varepsilon)$. Moreover, our assumptions are that the points are bounded away from $\partial\Omega$ uniformly in ε, and that $\deg(u, \partial B(a_i, R_0\varepsilon)) = 1$ for every ε, i.

From the blow-up analysis of Proposition 3.12 and assuming we are in the Coulomb gauge, the function $u_\varepsilon(a_i + \varepsilon y)$ converges modulo a subsequence and in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ to a solution v of (3.12) and respectively $\varepsilon A_\varepsilon(a_i + \varepsilon y)$ to 0. Moreover, as in the proof of Theorem 5.4 and using the upper bound $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le C|\log\varepsilon|$, we deduce from Theorem 5.2 that

$$\int_{\mathbb{R}^2} \left(1 - |v|^2\right)^2 < +\infty,$$

and from our hypothesis v must be of degree 1. Therefore modulo a translation and multiplication by a complex number of modulus one, v is equal to u_0, in view of Theorem 3.2, while $\varepsilon A'_\varepsilon(a_i + \varepsilon y)$ converges in C^1_{loc} to 0. Shifting the points a_i^ε by an order $O(\varepsilon)$, we may cancel the translation and find that there exist complex numbers $\{\lambda_i\}_i$ of modulus one such that $u_\varepsilon(a_i + \varepsilon.) - \lambda_i u_0$ converges to 0 in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ for any i. Thus

$$\lim_{\varepsilon\to 0} u_\varepsilon(a_i + \varepsilon.) - \lambda_i u_0 = \lim_{\varepsilon \to 0} \varepsilon A'_\varepsilon(a_i + \varepsilon.) = 0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{10.16}$$

where we recall that we are in the Coulomb gauge.

Now we fix $R > R_0$ and let $B_i = B(a_i, R\varepsilon)$.

Lower Bound for the Energy Inside the Balls

From (10.16), we have

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_{B_i}|\nabla_{A'_\varepsilon}u_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2=\frac{1}{2}\int_{B(0,R)}|\nabla u_0|^2+\frac{(1-|u_0|^2)^2}{2}. \tag{10.17}$$

This bounds the energy inside the balls.

Lower Bound Outside of the Balls

We follow the method of [61]. Define $\tilde{\Omega}=\Omega\setminus\cup_i B_i$, let $\rho=|u_\varepsilon|$ and φ be such that $u_\varepsilon=\rho e^{i\varphi}$. The function φ is well defined modulo 2π in $\tilde{\Omega}$ since $|u_\varepsilon|$ does not vanish there. Let us then define j by

$$j=\nabla\varphi-A'_\varepsilon-\nabla^\perp\Phi_\varepsilon \tag{10.18}$$

in $\tilde{\Omega}$, where Φ_ε is defined in (10.3).

To estimate $F_\varepsilon(u_\varepsilon,A'_\varepsilon,\tilde{\Omega})$ we write $\nabla\varphi-A'=j+\nabla^\perp\Phi$, where we have dropped the subscript ε for A' and Φ, and note that $-\operatorname{curl}A'=\Delta\Phi+\operatorname{curl}j=\Phi+\operatorname{curl}j$ in $\tilde{\Omega}$. It follows that

$$\int_{\tilde{\Omega}}\rho^2|\nabla\varphi-A'|^2+|\operatorname{curl}A'|^2=$$

$$\int_{\tilde{\Omega}}\rho^2|\nabla\Phi|^2+|\Phi|^2+\rho^2|j|^2+|\operatorname{curl}j|^2+2\rho^2\nabla^\perp\Phi\cdot j+2\Phi\operatorname{curl}j. \tag{10.19}$$

On the one hand

$$\int_{\tilde{\Omega}}(1-\rho^2)|\nabla\Phi|^2\le\varepsilon\left(\int_{\tilde{\Omega}}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{\frac{1}{2}}\left(\int_{\tilde{\Omega}}|\nabla\Phi|^4\right)^{\frac{1}{2}},$$

and from (10.8) we have,

$$|\nabla\Phi(x)|\le\frac{C}{\min_i|x-a_i|}. \tag{10.20}$$

Therefore
$$\int_{\tilde\Omega} |\nabla\Phi|^4 \le C \int_{R\varepsilon}^{1} \frac{2\pi r\, dr}{r^4} \le \frac{C}{R^2\varepsilon^2}.$$

Thus
$$\int_{\tilde\Omega} (\rho^2-1)|\nabla\Phi|^2 = \frac{C}{R}\left(\int_{\tilde\Omega} \frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{\frac12}. \tag{10.21}$$

Similarly, using the fact that $\|\nabla\Phi\|_{L^\infty(\tilde\Omega)} \le C/(R\varepsilon)$, which is deduced from (10.20), we have

$$\begin{aligned}\left|\int_{\tilde\Omega}(\rho^2-1)\nabla^\perp\Phi\cdot j\right| &\le \varepsilon\|\nabla\Phi\|_{L^\infty(\tilde\Omega)}\left(\int_{\tilde\Omega}\frac{(1-\rho^2)^2}{\varepsilon^2}\right)^{\frac12}\left(\int_{\tilde\Omega}|j|^2\right)^{\frac12}\\ &\le \frac{C}{R}\left(\int_{\tilde\Omega}\frac{(1-\rho^2)^2}{\varepsilon^2}+\int_{\tilde\Omega}|j|^2\right).\end{aligned}$$

Therefore in view of (10.19), and choosing R large enough we have, absorbing the terms in (10.21)–(10.22),

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \tilde\Omega) \ge \frac12 \int_{\tilde\Omega} |\nabla\Phi|^2 + |\Phi|^2 + 2\nabla^\perp\Phi\cdot j + 2\Phi\,\mathrm{curl}\, j. \tag{10.22}$$

On the other hand, integrating by parts, we have

$$\int_{\tilde\Omega} \nabla^\perp\Phi\cdot j + \Phi\,\mathrm{curl}\, j = \sum_i \int_{\partial B_i} \Phi j\cdot\tau. \tag{10.23}$$

But, denoting by $\overline{\Phi}$ the average of Φ on ∂B_i we have, in view of (10.18),

$$\int_{\partial B_i} \Phi j\cdot\tau = \int_{\partial B_i} (\Phi-\overline{\Phi}) j\cdot\tau + \int_{\partial B_i} \overline{\Phi}\left(\frac{\partial\varphi}{\partial\tau} + \frac{\partial\Phi}{\partial\nu} - A'\cdot\tau\right). \tag{10.24}$$

First of all, from the convergence of (10.7) we have $\|\Phi - \overline{\Phi}\|_{L^\infty(\partial B_i)} = o(1)$. Also, in view of (10.18) and Lemma 3.4, using the fact that $\rho \geq 1/2$ outside the balls, and combining (10.7) with (10.16) we find that

$$|j \cdot \tau| \leq \frac{1}{2}|\tau \cdot \nabla_{A'} u| + |\nabla \Phi| \leq \frac{C}{R\varepsilon}$$

on ∂B_i. It follows that

$$\int_{\partial B_i} (\Phi - \overline{\Phi}) j \cdot \tau = o(1). \tag{10.25}$$

Second of all, using (10.3) and keeping in mind that $d_i = 1$, we have that

$$\int_{\partial B_i} \left(\frac{\partial \varphi}{\partial \tau} + \frac{\partial \Phi}{\partial \nu} - A' \cdot \tau \right) = 2\pi + \int_{B_i} \Phi - 2\pi - \int_{B_i} \operatorname{curl} A'. \tag{10.26}$$

But in view of (10.7), we have $|\Phi| \leq C|\log \varepsilon|$ in B_i and from the energy upper bound $F_\varepsilon(u, A') \leq C|\log \varepsilon|$ and the Cauchy–Schwarz inequality we find

$$\left| \int_{B_i} \operatorname{curl} A' \right| \leq CR\varepsilon |\log \varepsilon|^{\frac{1}{2}}.$$

Replacing in (10.26) we get

$$\left| \overline{\Phi} \int_{\partial B_i} \frac{\partial \varphi}{\partial \tau} + \frac{\partial \Phi}{\partial \nu} - A' \cdot \tau \right| \leq C|\log \varepsilon| \left(|\log \varepsilon| R^2 \varepsilon^2 + R\varepsilon |\log \varepsilon|^{\frac{1}{2}} \right) = o(1).$$

Together with (10.25) this implies that the integral in (10.24) is $o(1)$ and then, in view of (10.22)–(10.23) that

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \tilde{\Omega}) \geq \int_{\tilde{\Omega}} |\nabla \Phi|^2 + |\Phi|^2 + o(1).$$

Then, using the same calculation as in (10.10)–(10.11) and adding (10.17), we find

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \Omega) \geq \pi n \log \frac{1}{R\varepsilon} + W(a_1, \dots, a_n) + \frac{n}{2} \int_{B(0,R)} |\nabla u_0|^2 + \frac{(1 - |u_0|^2)^2}{2} + o(1).$$

Since this is true for any R and since

$$\lim_{R\to+\infty} \frac{1}{2} \int_{B(0,R)} |\nabla u_0|^2 + \frac{(1-|u_0|^2)^2}{2} - (\pi \log R + \gamma) = 0$$

we obtain (10.15).

Bibliographic notes on Chapter 10: This chapter uses the tools and concepts (renormalized energy, canonical harmonic map, energy of the radial solution) used in the analysis of (1.2) for bounded numbers of vortices, by Bethuel–Brezis–Hélein and Brezis–Merle–Rivière in [43, 61], extended by Comte–Mironescu in [77, 79] and finally extended to the case with magnetic field by Bethuel–Rivière and Serfaty in [52, 181].

The specificity of the results presented in this chapter comes from the fact that the vortex-locations may depend on ε and may be very close to each other (due to the magnetic field), whereas previous results were expansions of the energy with respect to the limiting positions of the vortices. In this respect, these results are new.

Chapter 11

Branches of Solutions

In this chapter, we establish the existence of multiple branches of stable solutions of (GL) which have an arbitrary number of vortices n, with both n bounded and n unbounded, but not too large, in a wide regime of applied fields. These solutions are obtained by minimizing the energy G_ε over subsets U_n of the functional space which correspond, very roughly speaking, to configurations with n vortices (or only allow for such when minimizing); the heart of the matter consists in proving that the minimum is achieved in the interior of U_n, thus yielding locally minimizing solutions of the equations. These solutions turn out to be global energy minimizers in some narrow intervals of values of h_{ex}.

The setting is as in Chapter 9, i.e., we assume ξ_0 achieves its minimum at a unique point $p \in \Omega$ and $Q(x) = \langle D^2\xi_0(p)x, x\rangle$ is a positive definite quadratic form.

11.1 The Renormalized Energy w_n

Since Q is positive definite, we may write

$$Q(x) \geq c_0|x|^2 \tag{11.1}$$

for some positive constant c_0.

Given $n \in \mathbb{N}$, we introduce the "renormalized energy" w_n defined on $(\mathbb{R}^2)^n$ by

$$\boxed{w_n(x_1, \ldots, x_n) = -\pi \sum_{i \neq j} \log|x_i - x_j| + \pi n \sum_{i=1}^{n} Q(x_i)}. \tag{11.2}$$

Note that $w_0 = 0$ and $w_1(x) = \pi Q(x)$. In all that follows, we will also use the convention that $o(n^2)$ and $O(n^2)$ mean $o(1)$ and $O(1)$ when $n = 0$. It is clear that w_n tends to $+\infty$ as two x_i's tend to each other, or as any of the x_i's tends to $+\infty$. The minimum of w_n is achieved. The function w_n will correspond to the effective interaction between vortices which governs their positions. It contains two terms of opposite effects: the first term is a repulsive term, the second is a (quadratic) confinement term. For some results on the minimization of w_n, we refer again to [105] and Fig. 1.4.

Proposition 11.1 (Γ-convergence of w_n). *We have*

$$\frac{w_n}{n^2} \xrightarrow{\Gamma} I \quad \text{as } n \to \infty$$

(where I was introduced in (9.12)), in the sense that

1. *For every n-tuple of points $(a_1^n, \dots, a_n^n)$ such that $w_n(a_1^n, \dots, a_n^n) \le Cn^2$, up to extraction $\frac{1}{n}\sum_{i=1}^n \delta_{a_i^n} \rightharpoonup \mu$ in the narrow sense of measures and*

$$\liminf_{n\to\infty} \frac{w_n(a_1^n, \dots, a_n^n)}{n^2} \ge I(\mu). \tag{11.3}$$

2. *For every measure $\mu \in \mathcal{P}$ such that $I(\mu) < \infty$, there exist families of points $a_1^n, \dots, a_n^n$ such that $\frac{1}{n}\sum_{i=1}^n \delta_{a_i^n} \rightharpoonup \mu$ in the narrow sense of measures and*

$$\limsup_{n\to\infty} \frac{w_n(a_1^n, \dots, a_n^n)}{n^2} \le I(\mu).$$

An immediate consequence is:

Corollary 11.1.

$$\frac{\min_{(\mathbb{R}^2)^n} w_n}{n^2} \to I_0 = \min_{\mathcal{P}} I \quad \text{as } n \to +\infty,$$

and if $a_i^n, \dots, a_n^n$ minimize w_n, then

$$\frac{1}{n}\sum_{i=1}^n \delta_{a_i^n} \rightharpoonup \mu_0$$

where μ_0 is the minimizer of I described in Chapter 9.

Proof of the proposition. The upper bound 2 closely follows the proof of Proposition 7.4, hence we leave it to the reader.

Conversely, assume $a_1^n, \dots, a_n^n$ are such that $w_n(a_1^n, \dots, a_n^n) \le Cn^2$. Then the measures

$$\mu_n := \frac{1}{n} \sum_i \delta_{a_i^n}$$

converge weakly, up to extraction, to a measure μ. It remains to prove that the convergence is narrow, i.e., that μ is a probability measure, and that (11.3) holds. We start by proving the latter, assuming narrow convergence.

We begin by noting that, denoting by Δ the diagonal in $\mathbb{R}^2 \times \mathbb{R}^2$ and by Δ^c its complement we have

$$\frac{1}{n^2} w_n(a_1^n, \dots, a_n^n) = -\pi \iint_{\Delta^c} \log|x-y| \, d\mu_n(x) \, d\mu_n(y) + \pi \int Q(x) \, d\mu_n(x).$$

Moreover, since μ_n is a probability measure, this may be rewritten as

$$\frac{1}{n^2} w_n(a_1^n, \dots, a_n^n) = \pi \iint_{\Delta^c} -\log|x-y| d\mu_n(x) \, d\mu_n(y) + \frac{\pi}{2} \iint (Q(x) + Q(y)) \, d\mu_n(x) \, d\mu_n(y). \quad (11.4)$$

Now fix $M > 0$ and $R > 0$, and let $-\log^M(t) = \min(-\log t, M)$. Then $-\log^M$ is continuous in $K_R = [-R, R] \times [-R, R]$ and therefore

$$\lim_{n \to +\infty} \iint_{K_R \times K_R} -\log^M |x-y| + \frac{1}{2} (Q(x) + Q(y)) \, d\mu_n(x) \, d\mu_n(y) = \iint_{K_R \times K_R} -\log^M |x-y| + \frac{1}{2} (Q(x) + Q(y)) \, d\mu(x) \, d\mu(y).$$

Note that if R is larger than some R_0 and since Q is positive definite, the function $(x, y) \mapsto -\log^M |x-y| + (Q(x)+Q(y))/2$ is positive outside $K_R \times K_R$, and of course less than $-\log|x-y| + (Q(x) + Q(y))/2$. We deduce that

$$\pi \iint_{\Delta^c} -\log|x-y| d\mu_n(x)\,d\mu_n(y) + \frac{\pi}{2}\iint (Q(x)+Q(y))\; d\mu_n(x)\,d\mu_n(y)$$

$$\geq \pi \iint_{K_R\times K_R\setminus\Delta} -\log^M|x-y| d\mu_n(x)\,d\mu_n(y)$$

$$+\frac{\pi}{2}\iint_{K_R\times K_R} (Q(x)+Q(y))\; d\mu_n(x)\,d\mu_n(y)$$

$$= \pi \iint_{K_R\times K_R} \left(-\log^M|x-y| + \frac{1}{2}(Q(x)+Q(y))\right) d\mu_n(x)\,d\mu_n(y) - \frac{\pi M}{n}.$$

It follows that for any $M>0$ and any $R>R_0$ we have

$$\liminf_{n\to+\infty} \frac{1}{n^2} w_n(a_1^n,\dots,a_n^n)$$

$$\geq \pi \iint_{K_R\times K_R} \left(-\log^M|x-y| + \frac{1}{2}(Q(x)+Q(y))\right) d\mu(x)\,d\mu(y).$$

Taking the supremum over $R>R_0$ and M yields (11.3), noting that, since we have assumed narrow convergence, the measure μ is a probability measure and again

$$\iint \frac{1}{2}(Q(x)+Q(y))\; d\mu(x)\,d\mu(y) = \int Q(x)\,d\mu(x).$$

It remains to prove narrow convergence, but this is an easy consequence of the expression (11.4) for w_n. Using $-\log|x-y| \geq -\log(2\max(|x|,|y|))$ and (11.1), we find

$$-\log|x-y| + \frac{1}{2}(Q(x)+Q(y)) \geq -\log(2\|(x,y)\|_\infty) + \frac{c_0}{2}\|(x,y)\|_\infty^2,$$

and there exists R_0 such that if $\|(x,y)\|_\infty > R_0$, then the right-hand side is greater than $(c_0/4)\|(x,y)\|_\infty^2$, in particular positive. Now, given $R>R_0$, splitting $\mathbb{R}^2\times\mathbb{R}^2$ into $K_{R_0}\times K_{R_0}$, $(K_R\times K_R)^c$ and $(K_R\times K_R)\setminus(K_{R_0}\times K_{R_0})$ and denoting by k_R the number of couples (a_i^n,a_j^n) not belonging to $K_R\times K_R$, we deduce that

$$w_n(a_1^n,\dots,a_n^n) \geq -\pi n^2\log(2R_0) + \pi k_R \frac{c_0}{4} R^2,$$

and then dividing by n^2, that $k_R/n^2 \leq C/R^2$, where C does not depend on n or R. Therefore $\mu_n(\mathbb{R}^2 \setminus K_R) \leq C/R$, which implies the claimed narrow convergence. $\square$

The following will be useful:

Lemma 11.1. *Assume $h_{ex}(\varepsilon)$ and $n(\varepsilon)$ are such that $n \ll h_{ex}$ as $\varepsilon \to 0$ and let ξ_0 be defined by (7.2). Then we have, as $\varepsilon \to 0$,*

$$\inf_{\Omega^n} \left(-\pi \sum_{i \neq j} \log |a_i - a_j| + 2\pi h_{ex} \sum_i (\xi_0(a_i) - \underline{\xi_0}) \right) = \frac{\pi}{2}(n^2 - n) \log \frac{h_{ex}}{n} + \min_{(\mathbb{R}^2)^n} w_n + o(n^2).$$

Moreover, if n is assumed to be independent of ε and $\{(a_1^\varepsilon, \dots, a_n^\varepsilon)\}_\varepsilon$ are n-tuples of points such that

$$- \pi \sum_{i \neq j} \log |a_i^\varepsilon - a_j^\varepsilon| + 2\pi h_{ex} \sum_i (\xi_0(a_i^\varepsilon) - \underline{\xi_0}) = \pi \frac{n^2 - n}{2} \log \frac{h_{ex}}{n} + \min_{(\mathbb{R}^2)^n} w_n + o(n^2),$$

then, letting

$$\ell = \sqrt{\frac{n}{h_{ex}}}, \quad \tilde{a}_i^\varepsilon = \frac{a_i^\varepsilon - p}{\ell},$$

the n-tuple $(\tilde{a}_i^\varepsilon, \dots, \tilde{a}_n^\varepsilon)$ converges to a minimizer of w_n as $\varepsilon \to 0$.

Proof. For any n-tuple $(a_1, \dots, a_n)$, and letting $\tilde{a}_i = (a_i - p)/\ell$, we have

$$-\pi \sum_{i \neq j} \log |a_i - a_j| = -\pi(n^2 - n) \log \ell - \pi \sum_{i \neq j} \log |\tilde{a}_i - \tilde{a}_j| .$$

Moreover, writing a Taylor expansion of ξ_0 around its minimum point p we have

$$\xi_0(a_i) = \xi_0(p + \ell \tilde{a}_i) = \xi_0(p) + \frac{\ell^2}{2} Q(\tilde{a}_i) + o\left(\frac{n|\tilde{a}_i|^2}{h_{\text{ex}}} \right).$$

Combining the two relations, we find

$$-\pi\sum_{i\neq j}\log|a_i-a_j|+2\pi h_{\mathrm{ex}}\sum_i(\xi_0(a_i)-\underline{\xi_0})$$
$$=\pi(n^2-n)\log\frac{1}{\ell}+w_n(\tilde a_1,\dots,\tilde a_n)+o\left(n\sum_i|\tilde a_i|^2\right),$$

from which the result easily follows. □

11.2 Branches of Solutions

We now consider $h_{\mathrm{ex}}\geq 0$, N, an integer, and $\varepsilon>0$, and we try to show the existence of solutions to the Ginzburg–Landau equations with the given parameters $h_{\mathrm{ex}},\varepsilon$ which have the prescribed number of vortices N. These solutions will be obtained by minimizing the Ginzburg–Landau energy among configurations with N vortices. We let

$$L=\sqrt{\frac{N}{h_{\mathrm{ex}}}}. \tag{11.5}$$

Note that from the results of Chapter 9 we expect L to be the typical distance between the vortices and p for our solutions.

Let us state precisely the conditions under which we will be able to show the existence of such solutions.

Definition 11.1. *We say $h_{ex}(\varepsilon),N(\varepsilon)$ are* admissible *if the following conditions hold.*

1. *There exists $\alpha_0<1/2$ such that $h_{ex}<\varepsilon^{-\alpha_0}$.*
2. *If $N\neq 0$, then*

$$N^2\leq\eta h_{ex},\quad N^2\log\frac{1}{L}\leq\eta\log\frac{L}{\varepsilon}, \tag{11.6}$$

for some η small enough depending on Ω and α_0, to be specified later.

Several remarks can be made on the definition above.

- Writing the second relation in (11.6) as $(N^2+\eta)\log\frac{1}{L}\leq\eta\log\frac{1}{\varepsilon}$ and replacing L by its definition, we find that (11.6) is equivalent to
$$N^2\leq\eta h_{\mathrm{ex}},\quad h_{\mathrm{ex}}\leq N\varepsilon^{-\frac{2\eta}{N^2+\eta}}.$$

- If we assume $h_{\text{ex}} < C|\log\varepsilon|$, (11.6) is satisfied for example if $N^2 \leq \eta h_{\text{ex}}$ and $1 \leq N \leq |\log\varepsilon|^\gamma$, for some $\gamma < 1/2$. This is seen by noting that in this case and from the very definition of L we have $C|\log\varepsilon|^{-1/2} \leq L \leq C$, and plugging this into (11.6).

- Since for $N \neq 0$ we always have $N \geq 1$, we deduce from (11.6) that $L^2 \leq \eta/N \leq \eta$. Thus also, $N^2 \log(N/\eta) \leq 2\eta \log(L/\varepsilon)$. Using again that L is bounded we find $N^2 \log(N/\eta) \leq C|\log\varepsilon|$ which implies

$$N \ll \sqrt{|\log\varepsilon|}.$$

We will distinguish three cases in our proofs. Firstly the case where L does not tend to zero. In this case, after extraction, $N = O(1)$ and $h_{\text{ex}} = O(1)$. Indeed if $h_{\text{ex}} \to +\infty$, then (11.6) implies $N \leq \sqrt{\eta h_{\text{ex}}} \ll h_{\text{ex}}$, contradicting the assumption, hence we may assume that h_{ex} remains bounded and thus N from (11.6) also. Secondly the case where L tends to zero and $N = O(1)$. The vortices then concentrate around p but their number is bounded independently of ε. The last case is the one where L tends to zero and $N \to \infty$.

Our main result is the following:

Theorem 11.1 (Branches of stable solutions). *Given $\alpha_0 \in (0, 1/2)$, choosing η small enough depending on Ω and α_0, and given admissible $N(\varepsilon)$ and $h_{ex}(\varepsilon)$ in the sense of Definition 11.1, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists $(u_\varepsilon, A_\varepsilon)$ with the following properties.*

First, $(u_\varepsilon, A_\varepsilon)$ is a locally minimizing critical point of G_ε hence a stable solution of (GL). Also, u_ε has exactly N zeroes $a_1^\varepsilon, \ldots, a_N^\varepsilon$ and there exists $R > 0$ such that $|u_\varepsilon| \geq \frac{1}{2}$ in $\Omega \setminus \cup_i B(a_i^\varepsilon, R\varepsilon)$ and $\deg(u_\varepsilon, \partial B(a_i^\varepsilon, R\varepsilon)) = 1$. Moreover, the following holds.

1. *If N and h_{ex} are independent of ε, then, possibly after extraction, the n-tuple $(a_1^\varepsilon, \ldots, a_N^\varepsilon)$ converges as $\varepsilon \to 0$ to a minimizer of $R_{N,h_{ex}}$, where*

$$R_{N,h_{ex}}(x_1, \ldots, x_N) = -\pi \sum_{i \neq j} \log|x_i - x_j| \\ + \pi \sum_{i,j} S_\Omega(x_i, x_j) + 2\pi h_{ex} \sum_i \xi_0(x_i),$$

and defining γ by (3.15),

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = {h_{ex}}^2 J_0 + \pi N |\log\varepsilon| + \min_{\Omega^N} R_{N,h_{ex}} + N\gamma + o(1) \quad as\ \varepsilon \to 0.$$

2. *If N is independent of ε and $h_{ex} \to +\infty$, then, possibly after extraction and letting $\tilde{a}_i^\varepsilon = (a_i^\varepsilon - p)/\ell$, the n-tuple $(\tilde{a}_1^\varepsilon, \dots, \tilde{a}_N^\varepsilon)$ converges as $\varepsilon \to 0$ to a minimizer of*

$$w_N(x_1, \dots, x_N) = -\pi \sum_{i \neq j} \log|x_i - x_j| + \pi N \sum_{i=1}^N Q(x_i),$$

   and defining $f_\varepsilon(N)$ by (9.4),

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(N) + \min_{(\mathbb{R}^2)^N} w_N + N\gamma + o(1), \quad as\ \varepsilon \to 0. \quad (11.7)$$

3. *If $N \to \infty$ and $h_{ex} \to +\infty$, then, defining $\tilde{a}_i^\varepsilon$ as above, we have*

$$\frac{1}{N}\sum_{i=1}^N \delta_{\tilde{a}_i^\varepsilon} \rightharpoonup \mu_0, \quad G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(N) + N^2 I_0 + o(N^2), \quad as\ \varepsilon \to 0$$

   where μ_0 is the unique minimizer of I, $I_0 = I(\mu_0)$ and where the convergence is in the narrow sense of measures.

4. *In the case $N = 0$, we have $|u_\varepsilon| \to 1$ in $L^\infty(\Omega)$. (This is called the Meissner solution.)*

Remark 11.1. *There is a sort of continuity between the multiple renormalized energies found above, since w_n can be considered as a limit of $R_{n,h_{ex}}$ when $h_{ex} \to +\infty$, and I as a limit of w_n as $n \to \infty$, as seen in Proposition 11.1.*

11.3 The Local Minimization Procedure

We introduce the following sets

$$U_0 = \{(u, A) \in X_\Omega \mid F_\varepsilon(u, A') < \varepsilon^q\}, \quad (11.8)$$

where X_Ω was defined in (3.2), $A' = A - h_{\text{ex}}\nabla^\perp h_0$ and $q \in (0,1)$ is to be specified later. For $N \geq 1$, we let

$$U_N = \left\{(u, A) \in X_\Omega, |F_\varepsilon(u, A') - f_\varepsilon^0(N)| < BN^2\right\}, \quad (11.9)$$

where $B > 0$ is a constant to be determined later, and f_ε^0 is as in (9.10), i.e., $f_\varepsilon^0(N) = \pi N \log \frac{L}{\varepsilon} + \pi N^2 \log \frac{1}{L} + \pi N^2 S_\Omega(p,p)$.

The branches of solutions will be obtained by minimizing the Ginzburg–Landau energy in U_N. We will first show that the infimum is achieved, and then that it is a critical point. The rest of the statements in the theorem will follow rather easily. In the course of the proof we will define the constants η, q and B in (11.6), (11.8) and (11.9).

11.4 The Case $N = 0$

We prove the theorem in this particularly simple case. Consider a minimizing sequence $\{(u_n, A_n)\}_n$ in U_0. Arguing as in Proposition 3.5 we may assume that A_n satisfies the Coulomb gauge condition and deduce that the sequence is bounded in $H^1 \times H^1$. Then a subsequence converges weakly to (u, A). Arguing again as in Proposition 3.5, we have $F_\varepsilon(u, A') \le \liminf_n F_\varepsilon(u_n, A_n')$ hence $(u, A) \in \overline{U}_0$ and $G_\varepsilon(u, A) \le \liminf_n G_\varepsilon(u_n, A_n)$ hence (u, A) is a minimizer of G_ε over $\overline{U}_0$.

To prove that $(u, A) \in U_0$, we consider the test-configuration $(1, h_{\rm ex}\nabla^\perp\xi_0)$. It belongs to U_0 since $F_\varepsilon(1, h_{\rm ex}\nabla^\perp\xi_0 - h_{\rm ex}\nabla^\perp\xi_0) = F_\varepsilon(1, 0) = 0$. Thus $\inf_{U_0} G_\varepsilon \le G_\varepsilon(1, h_{\rm ex}\nabla^\perp\xi_0) = {h_{\rm ex}}^2 J_0$. Since $(u, A) \in \overline{U}_0$, we have $F_\varepsilon(u, A') \le \varepsilon^q$ with $q > 0$, and we may apply Theorem 4.1 with $r = \varepsilon^{\frac{q+1}{2}}$. The upper bound $F_\varepsilon(u, A') \le \varepsilon^q$ implies that if ε is small enough, the degrees of the balls we obtain are all equal to zero. Thus, applying Theorem 6.1, we find

$$h_{\rm ex}\|\mu(u, A')\|_{(C^{0,1}(\Omega))^*} \le C h_{\rm ex} \varepsilon^{\frac{q+1}{2}} F_\varepsilon(u, A') \le C\varepsilon^{\frac{3q}{2}},$$

where we have again used the bound on $F_\varepsilon(u, A')$ together with the assumption $h_{\rm ex} \le \varepsilon^{-\alpha_0}$, with $\alpha_0 < 1/2$. Applying the energy-decomposition lemma, Lemma 7.3, we obtain

$$G_\varepsilon(u, A) = {h_{\rm ex}}^2 J_0 + F_\varepsilon(u, A') + O\left(\varepsilon^{\frac{3q}{2}} + \varepsilon^{1-2\alpha_0+\frac{q}{2}}\right).$$

Since $\alpha_0 < \frac{1}{2}$ we may choose $q \in (0, 1)$ such that $1 - 2\alpha_0 + q/2 > q$ and then

$$G_\varepsilon(u, A) = {h_{\rm ex}}^2 J_0 + F_\varepsilon(u, A') + o(\varepsilon^q).$$

But $G_\varepsilon(u, A) \le {h_{\rm ex}}^2 J_0$ therefore $F_\varepsilon(u, A') = o(\varepsilon^q)$ and $(u, A) \in U_0$ for ε small enough.

Now, since U_0 is open in $H^1 \times H^1$, the minimizer (u, A) must be a critical point of G_ε. Then we know from Corollary 3.1 that $|\nabla|u|| \leq |\nabla_A u| \leq C/\varepsilon$ and together with the fact that

$$\frac{1}{2}\int_\Omega |\nabla|u||^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \leq F_\varepsilon(u, A') = o(1),$$

this implies by standard arguments (see [43], Theorem III.1) that $\||u|-1\|_{L^\infty} = o(1)$ as $\varepsilon \to 0$. This completes the proof of the theorem in the case $N = 0$.

11.5 Upper Bound for $\inf_{U_N} G_\varepsilon$

We prove:

Proposition 11.2. *There exists $B_0, L_0 > 0$ depending only on Ω such that the following holds.*

Assume $N(\varepsilon), h_{ex}(\varepsilon)$ are such that $L < L_0$ and $h_{ex} \ll \frac{1}{\varepsilon}$, where L is defined in (11.5). *Assume in addition, if N tends to $+\infty$ as $\varepsilon \to 0$, that $N \ll h_{ex}$. (All these conditions are satisfied in particular for admissible h_{ex} and N.) Then, defining U_N by* (11.9) *with $B > B_0$ we have, if ε is small enough,*

$$\inf_{U_N} G_\varepsilon \leq f_\varepsilon(N) + B_0 N^2, \tag{11.10}$$

where f_ε is as in (9.4), i.e.,

$$f_\varepsilon(N) = {h_{ex}}^2 J_0 - 2\pi N h_{ex}|\underline{\xi_0}| + \pi N \log\frac{L}{\varepsilon} + \pi N^2 \log\frac{1}{L} + \pi N^2 S_\Omega(p,p).$$

Proof. This upper bound is obtained through constructions which we have already performed.

We begin with the case where N is independent of ε. We define for $0 \leq k \leq N$

$$a_k = e^{\frac{2ik\pi}{N}}, \quad a_k^\varepsilon = p + La_k,$$

where p is the minimum point of ξ_0 and L is defined in (11.5), i.e., equidistributed points on the circle of center p and radius L. If L is small enough depending on Ω, then for any ε the points $\{a_k^\varepsilon\}_k$ are inside Ω and bounded away from $\partial\Omega$. Moreover the distance between them is proportional to L/N which, under our hypotheses, is much greater than ε. Then from Proposition 10.1, there is a family of configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ satisfying (10.1)–(10.2) with n replaced by N and $d_i = 1$ for every $0 \le i \le N-1$.

Since $|a_i^\varepsilon - a_j^\varepsilon| = L|a_i - a_j|$, we have

$$\sum_{i\neq j} \log|a_i^\varepsilon - a_j^\varepsilon| = N(N-1)\log L + \sum_{k\neq l} \log\left|e^{\frac{2ik\pi}{N}} - e^{\frac{2il\pi}{N}}\right|.$$

But, given l, $\prod_{k\neq l}\left(e^{\frac{2ik\pi}{N}} - e^{\frac{2il\pi}{N}}\right) = \left|\prod_{k=1}^{N-1}(1 - e^{\frac{2ik\pi}{N}})\right|$ and using $\prod_{k=1}^{N-1}\left(x - e^{\frac{2ik\pi}{N}}\right) = \frac{x^N-1}{x-1} = 1 + x + \cdots + x^{N-1}$, we find that $\prod_{k\neq l}\left(e^{\frac{2ik\pi}{N}} - e^{\frac{2il\pi}{N}}\right) = N$ and thus that

$$\sum_{i\neq j} \log|a_i^\varepsilon - a_j^\varepsilon| = N(N-1)\log L + N\log N.$$

Moreover, since S_Ω is locally C^1 in Ω, we have

$$\left|\sum_{i,j} S_\Omega(a_i^\varepsilon, a_j^\varepsilon) - N^2 S_\Omega(p,p)\right| \le CLN^2,$$

where C denotes a generic constant depending only on Ω. Finally, we may write

$$h_{\mathrm{ex}}\xi_0(a_i^\varepsilon) = h_{\mathrm{ex}}\underline{\xi_0} + h_{\mathrm{ex}}L^2\frac{\xi_0(p+La_i) - \xi_0(p)}{L^2} \le h_{\mathrm{ex}}\underline{\xi_0} + CN,$$

since $\nabla\xi_0(p) = 0$ and $h_{\mathrm{ex}}L^2 = N$, where C is another constant depending only on Ω.

The above, together with (10.1)–(10.2) yield for ε small enough (in view of the definitions of f_ε and f_ε^0)

$$\left|F_\varepsilon(u_\varepsilon, A'_\varepsilon) - f_\varepsilon^0(N)\right| < B_0 N^2,$$

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le f_\varepsilon(N) + B_0 N^2,$$

with B_0 a constant depending only on Ω. The first inequality ensures that if B is chosen large enough depending on Ω, then $(u_\varepsilon, A_\varepsilon) \in U_N$ and then the second inequality implies (11.10).

The case where N tends to $+\infty$ is similar. In this case we have $1 \ll N \ll h_{\mathrm{ex}}$ and $h_{\mathrm{ex}} \ll C/\varepsilon$. Then using Proposition 9.1, taking as the measure μ any fixed compactly supported probability measure such that $I(\mu) < +\infty$, we find that there exists a family of configurations $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ satisfying

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) = f^0_\varepsilon(N) + O(N^2) \tag{11.11}$$

and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(N) + N^2 I(\mu) + o(N^2). \tag{11.12}$$

The relation (11.11) immediately yields that if B is large enough, then for ε small enough $(u_\varepsilon, A_\varepsilon) \in U_N$, and then using (11.12) that if $B_0 > I(\mu)$, then (11.10) is true for ε small enough. □

11.6 Minimizing Sequences Stay Away from ∂U_N

In this section we prove the "hard" analysis part of the proof that the minimum of G_ε over U_N is achieved.

Proposition 11.3. *Given $\alpha_0 \in (0, 1/2)$ we may choose $\eta > 0$ and $B > 0$ depending on Ω, α_0 such that if $N(\varepsilon), h_{ex}(\varepsilon)$ are admissible and U_N is defined by* (11.9)*, the following holds.*

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there exists $\delta_\varepsilon \in (0, 1)$ such that for any $(u, A) \in X_\Omega$,

$$G_\varepsilon(u, A) < \inf_{U_N} G_\varepsilon + 1 \quad \Longrightarrow \quad \mathrm{dist}\,((u, A), \partial U_N) > \delta_\varepsilon.$$

Here we use $d\,((u_1, A_1), (u_2, A_2)) = \|u_1 - u_2\|_{H^1} + \|A_1 - A_2\|_{H^1}$ as the distance in X_Ω.

Once this result is proved, it will follow from Ekeland's variational principle that for ε small enough there exists a minimizing sequence in U_N which is a Palais–Smale sequence bounded away from ∂U_N. Then this sequence will converge strongly to a minimizer of G_ε in U_N, hence a locally minimizing critical point of G_ε. The rest of this section is devoted to proving Proposition 11.3. Throughout the proof, we assume that $\alpha_0 \in (0, 1/2)$, that $N(\varepsilon), h_{\mathrm{ex}}(\varepsilon)$ are admissible and that $G_\varepsilon(u, A) < \inf_{U_N} G_\varepsilon + 1$.

We already noted that (11.6) implies that $N \ll \sqrt{|\log \varepsilon|}$ as $\varepsilon \to 0$, for any choice η. Thus the very definition of U_N implies that if $(u, A) \in U_N$, then $F_\varepsilon(u, A') = O(|\log \varepsilon|^2)$ as $\varepsilon \to 0$. In particular for any $\alpha \in (0, 1)$, if ε is small enough, then $F_\varepsilon(u, A') < \varepsilon^{\alpha-1}$. Since $h_{\text{ex}} \le \varepsilon^{-\alpha_0}$ for some $\alpha_0 \in (0, 1/2)$, it follows that (9.5) is satisfied for *any* $\alpha \in [\alpha_0, 1)$. In particular we may choose α such that $1 + \alpha_0 < 3\alpha/2$ and $2\alpha_0 < \alpha$, implying

$$h_{\text{ex}}\varepsilon^{\frac{3\alpha}{2}-1} = o(1), \quad {h_{\text{ex}}}^2\varepsilon^\alpha = o(1). \tag{11.13}$$

This choice will prove useful below. Note that α is chosen depending on α_0, thus depending on α or α_0 means the same thing.

In any case, it follows from (9.5) that as in Chapter 9, if $(u, A) \in U_N$, we may associate to it a family of large balls $\mathcal{B}$ of total radius $r = \frac{1}{\sqrt{h_{\text{ex}}}}$ and total degree d defined by (9.7), and small balls $\mathcal{B}'$ of total radius and degree defined by (9.6). The first and elementary link between n, n' and N is given by the fact that $F_\varepsilon(u, A) \ge C_\alpha n' |\log \varepsilon|$, which follows from Theorem 4.1, together with $n' \ge n$ and $F_\varepsilon(u, A) \le CN|\log \varepsilon|$, which follow from the definition of $\mathcal{B}$ and $\mathcal{B}'$ and the definition of U_N respectively. It follows that

$$n \le n' \le C_\alpha N \le C_\alpha \sqrt{|\log \varepsilon|}, \tag{11.14}$$

where C_α is a constant depending only on α.

— *Step 1:* $n + \frac{\alpha}{2}(n' - n) \le N$. We improve (11.14). From the definition of U_N and (11.5) we have

$$F_\varepsilon(u, A') \le \pi N \log \frac{\sqrt{N}}{\varepsilon\sqrt{h_{\text{ex}}}} + \pi N^2 \log \frac{1}{L} + BN^2$$

while since (9.5) is satisfied, Proposition 9.3 applies and (9.25) yields

$$F_\varepsilon(u, A') \ge \pi n \log \frac{1}{\varepsilon n\sqrt{h_{\text{ex}}}} + \pi\frac{\alpha}{2}(n' - n)|\log \varepsilon| - Cn.$$

We divide the above inequalities by $D = \log \frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}$, which is greater than $\frac{3}{4}|\log \varepsilon|$. Noting that from (11.6) and (11.14), $(N \log N)/D$ and $(n \log n)/D$ are $o(1)$ as $\varepsilon \to 0$, we find

$$\pi\left(n + \frac{\alpha}{2}(n' - n)\right) \le \pi N + \frac{\pi N^2 \log \frac{1}{L} + BN^2}{\log \frac{1}{\varepsilon\sqrt{h_{\text{ex}}}}} + o(1). \tag{11.15}$$

If N and $h_{\rm ex}$ are bounded independently of ε, then the right-hand side is equal to $\pi N + o(1)$ and therefore for ε small enough we find $n+\frac{\alpha}{2}(n'-n) \le N$, as claimed.

If not, then $L = o(1)$ and therefore $N^2 \log \frac{1}{L} + BN^2 \sim N^2 \log \frac{1}{L}$. But from (11.6) we have

$$\frac{\pi N^2 \log \frac{1}{L}}{\log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}}} \sim \frac{\pi N^2 \log \frac{1}{L}}{\log \frac{L}{\varepsilon}} \le \pi\eta$$

and therefore, assuming

$$\eta < \alpha/2, \tag{11.16}$$

we deduce from (11.15) that if ε is small enough, then $n + \frac{\alpha}{2}(n'-n) < N + \alpha/2$, and thus $n + \frac{\alpha}{2}(n'-n) \le N$ in this case also. □

— *Step 2: $n = n' = N$, and the vortices are bounded away from ∂U_N.* Assuming

$$B > B_0, \quad L < L_0 \tag{11.17}$$

where B_0, L_0 are defined in Proposition 11.2, we have

$$G_\varepsilon(u, A) \le \inf_{U_N} G_\varepsilon + 1 \le f_\varepsilon(N) + B_0 N^2 + 1. \tag{11.18}$$

On the other hand, (9.24) in Proposition 9.3 yields

$$\begin{aligned} G_\varepsilon(u, A) \ge {h_{\rm ex}}^2 J_0 + 2\pi h_{\rm ex} \sum_i d_i \xi_0(b_i) \\ + F_\varepsilon(u, A') - C(n'-n) r h_{\rm ex} + o(1) \\ \ge {h_{\rm ex}}^2 J_0 + 2\pi h_{\rm ex} \sum_i d_i \xi_0(b_i) + \pi N \log \frac{L}{\varepsilon} \\ + \pi N^2 \log \frac{1}{L} - BN^2 - C(n'-n)\sqrt{h_{\rm ex}} + o(1), \end{aligned} \tag{11.19}$$

where we have bounded $F_\varepsilon(u, A')$ from below using the definition of U_N, and used (11.13). Comparing the above inequalities yields

$$\begin{aligned} 2\pi h_{\rm ex} \left(\sum_i d_i \xi_0(b_i) - N \underline{\xi_0} \right) \\ \le 2BN^2 + C(n'-n)\sqrt{h_{\rm ex}} + 1 + o(1). \end{aligned} \tag{11.20}$$

Using the fact that $1 \le N^2 \le \eta h_{\rm ex}$ and $n, n' \le CN$ (see (11.14)), and choosing

$$\eta < \frac{1}{4}\min\left(\frac{\pi|\underline{\xi_0}|}{B+1}, \frac{|\underline{\xi_0}|}{C}\right), \tag{11.21}$$

the right-hand side of (11.20) is bounded above for ε small enough by $\pi h_{\rm ex}|\underline{\xi_0}|$. But the function ξ_0 is negative and we know from Step 1 that $n = \sum |d_i| \le N$. Thus the left-hand side can be written as a sum of positive terms

$$2\pi h_{\rm ex}\sum_{d_i>0} d_i\left(\xi_0(b_i) - \underline{\xi_0}\right) + 2\pi h_{\rm ex}\sum_{d_i<0} d_i(\xi_0(b_i) + \underline{\xi_0}) + 2\pi h_{\rm ex}(N-n)|\underline{\xi_0}|,$$

and dividing (11.20) by $2\pi h_{\rm ex}$ thus yields

$$(N-n)|\underline{\xi_0}| + \sum_{d_i>0} d_i\left(\xi_0(b_i) - \underline{\xi_0}\right) + \sum_{d_i<0} d_i(\xi_0(b_i) + \underline{\xi_0}) \le \frac{1}{2}|\underline{\xi_0}|, \tag{11.22}$$

where all the terms are positive. It follows that, for ε small enough, $N - n \le \frac{1}{2}$ or $N \le n$ which together with Step 1 proves that $n = n' = N$, as claimed. In particular $\ell = L$. Moreover, from (11.22), for every i such that $d_i < 0$ we find $|\xi_0(b_i) + \underline{\xi_0}| \le \frac{1}{2}|\underline{\xi_0}|$, which is impossible, and for every i such that $d_i > 0$ we have $\xi_0(b_i) - \underline{\xi_0} \le \frac{1}{2}|\underline{\xi_0}|$. Therefore, we have shown that if $(u, A) \in X_\Omega$ is such that $G_\varepsilon(u, A) < \inf_{U_N} G_\varepsilon + 1$, then

$$d_i \ne 0 \quad\Longrightarrow\quad d_i > 0, \quad b_i \in \left\{x \in \Omega \mid \xi_0(x) \le \frac{1}{2}\underline{\xi_0}\right\}. \tag{11.23}$$

This implies that the vortices are bounded away from $\partial\Omega$ since ξ_0 vanishes there. □

— *Step 3: Conclusion.* We apply Proposition 9.4, choosing $K = K_0$ and $\delta = \delta_0$. It follows that if L is small enough depending on Ω then (9.40) holds. Together with (9.25) it implies that

$$F_\varepsilon(u, A') + 2\pi h_{\rm ex}\sum_i d_i\xi_0(b_i) \ge \pi N\log\frac{L}{\varepsilon} + \pi N^2\log\frac{1}{L} + 2\pi N h_{\rm ex}\underline{\xi_0} + R,$$

where, denoting C_Ω as a positive constant depending only on Ω,

$$R = -\pi\left(\frac{3N}{2}\log N + N^2\log\frac{K_0}{\delta_0} + N^2{\delta_0}^2 + \frac{N^{3/2}}{K_0} + CN\right) \ge -C_\Omega N^2.$$

Together with (9.24), we deduce from the above and (11.19) that

$$G_\varepsilon(u,A) \geq f_\varepsilon(N) - \pi S_\Omega(p,p)N^2 - C_\Omega N^2.$$

This lower bound matches the upper bound of (11.10) up to CN^2; therefore (9.42) in Proposition 9.4 is satisfied, and yields together with (9.25)

$$F_\varepsilon(u,A') \geq \pi N \log\frac{L}{\varepsilon} + \pi N^2 \log\frac{1}{L} - C_\Omega N^2. \tag{11.24}$$

From (9.24) we have $F_\varepsilon(u,A') \leq G_\varepsilon(u,A) - {h_{\mathrm{ex}}}^2 J_0 - 2\pi N h_{\mathrm{ex}} \underline{\xi_0} + o(1)$, which together with (11.18) implies

$$F_\varepsilon(u,A') \leq f_\varepsilon^0(N) + B_0 N^2 + 1 + o(1),$$

and thus

$$\left|F_\varepsilon(u,A') - f_\varepsilon^0(N)\right| \leq B_0 N^2 + 2 + C_\Omega N^2 + \pi S_\Omega(p,p)N^2. \tag{11.25}$$

If $(v,B) \in \partial U_N$, then since F_ε is continuous with respect to the distance we have chosen on X_Ω,

$$\left|F_\varepsilon(v,B') - f_\varepsilon^0(N)\right| = BN^2,$$

thus if we choose

$$B > C_\Omega + B_0 + \pi S_\Omega(p,p) + 3, \tag{11.26}$$

then we deduce from (11.24)–(11.25) that if $(v,B) \in \partial U_N$, then $|F_\varepsilon(v,B') - F_\varepsilon(u,A')| > N^2$. From the uniform continuity of F_ε in U_N this proves that

$$\operatorname{dist}\left((v,B),(u,A)\right) > \delta_\varepsilon,$$

for some number δ_ε which does not depend on the choice of (u,A) satisfying the hypothesis. Proposition 11.3 is proved, with B chosen large enough depending on Ω to satisfy (11.17), (11.26), and η chosen small enough as to satisfy (11.16), (11.17) and (11.21). Indeed, from the definition of L, we have $L^2 \leq \eta$, thus if η is small enough, then $L < L_0$ is satisfied. □

11.7 $\inf_{U_N} G_\varepsilon$ is Achieved

The rest of the proof that $\inf_{U_N} G_\varepsilon$ is achieved relies on rather well-known arguments. We recall Ekeland's variational principle (see for example [27]):

Ekeland's principle: Assume X is a metric space and $F : X \to \mathbb{R}\cup\{+\infty\}$ is a proper lower semicontinuous function bounded from below. Assume that for some integer n we are given x_n such that $F(x_n) \leq \inf_X F + 1/n$. Then for any $\lambda > 0$, there exists $\tilde{x}_n \in X$ such that

$$d(x_n, \tilde{x}_n) \leq \lambda, \quad F(\tilde{x}_n) \leq F(x_n),$$

and for every $x \in X$,

$$\frac{F(x) - F(\tilde{x}_n)}{d(x, \tilde{x}_n)} \geq \frac{1}{n\lambda}.$$

We apply this to the metric space U_N^c consisting of those $(u, A) \in U_N$ satisfying the Coulomb gauge condition, endowed with the distance function $d\left((u_1, A_1), (u_2, A_2)\right) = \|u_1 - u_2\|_{H^1} + \|A_1 - A_2\|_{H^1}$. The function $(u, A) \mapsto G_\varepsilon(u, A)$ is continuous, and even differentiable. We choose B, η as in Proposition 11.3, and ε small enough. We let $\lambda = \delta_\varepsilon/2$ and consider a sequence $\{(v_n, B_n)\}_n$ in U_N^c such that $G_\varepsilon(v_n, B_n) \leq \inf_{U_N} G_\varepsilon + 1/n$. Then from Ekeland's principle there exists a sequence $\{(u_n, A_n)\}_n$ in U_N^c such that $G_\varepsilon(u_n, A_n) \leq \inf_{U_N} G_\varepsilon + 1/n$ and such that $d\left((v_n, B_n), (u_n, A_n)\right) \leq \delta_\varepsilon/2$. Using Proposition 11.3, this implies that (u_n, A_n) remains at a distance at least $\delta_\varepsilon/2$ from ∂U_N. The last property of (u_n, A_n) given by Ekeland's principle implies, since G_ε is in fact C^1, that the norm of the differential of G_ε at (u_n, A_n) tends to 0 as $n \to +\infty$.

To summarize, $\{(u_n, A_n)\}_n$ is a minimizing sequence for G_ε in U_N which satisfies the Coulomb gauge condition, which remains bounded away from ∂U_N and which is a Palais–Smale sequence. It remains to show that such a sequence converges strongly in $H^1 \times H^1$. Its limit will belong to the interior of U_N and minimize G_ε, as claimed.

We now sketch the proof of strong convergence of Palais–Smale sequences. First $\{(u_n, A_n)\}_n$ is bounded in $H^1 \times H^1$ using the arguments in Proposition 3.5 because $G_\varepsilon(u_n, A_n)$ is bounded and (u_n, A_n) satisfies the Coulomb gauge condition. Thus, it has a subsequence which converges weakly to some (u, A) and $(DG_\varepsilon)_{(u_n,A_n)}(u - u_n, A - A_n)$ tends to zero: this is the Palais–Smale condition. This reads

$$\lim_{n\to+\infty}\Big(\int_\Omega \nabla_{A_n}u_n\cdot\nabla_{A_n}(u-u_n) - \frac{(1-|u_n|^2)}{2\varepsilon^2}u_n\cdot(u-u_n) + \int_\Omega \operatorname{curl}A_n\operatorname{curl}(A-A_n) + j_n\cdot(A-A_n)\Big) = 0,$$

where $j_n = (iu_n, \nabla_{A_n}u_n)$. Since A and A_n satisfy the Coulomb gauge condition we may replace curl by ∇ in the second integral. Then the weak H^1 convergence and strong L^q convergence (up to extraction), for $q > 0$, of both u_n and A_n allows us to deduce from the above that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^2 + |\nabla A_n|^2 = \int_\Omega |\nabla u|^2 + |\nabla A|^2.$$

On the other hand, by weak convergence, we have the inequalities $\int_\Omega |\nabla u|^2 \le \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_n|^2$ and $\int_\Omega |\nabla A|^2 \le \liminf_{\varepsilon\to 0}|\nabla A|^2$ hence there must be equality in each, which proves the strong convergence of the sequences, and the fact that $\inf_{U_N} G_\varepsilon$ is achieved.

11.8 Proof of Theorem 11.1

Until now we have shown under suitable hypothesis the existence of a minimizer $(u_\varepsilon, A_\varepsilon)$ for $\inf_{U_N} G_\varepsilon$, which is an interior point of U_N hence a local minimizer of G_ε and then a stable solution of (GL). This allows us to give a more detailed description of $(u_\varepsilon, A_\varepsilon)$.

$\{|u_\varepsilon| < 1/2\}$ is bounded away from $\partial\Omega$

Let $\omega = \{\xi_0 < \underline{\xi_0}/4\}$. Then, as seen in (11.23), the points b_i such that $d_i \neq 0$ are inside ω and bounded away from $\partial\Omega$, and $d_i > 0$. We apply Theorem 4.1 in ω to $(u_\varepsilon, A'_\varepsilon)$ with $r = L$, and call the resulting collection of balls the *new balls*. They satisfy (4.3). We claim their total degree D (sum of absolute values of the degrees) satisfies $D \ge N$. Indeed since the total radius of the old and new balls both go to zero as $\varepsilon \to 0$, we may find for each ε small enough a simple closed curve γ which is inside ω, which does not intersect the new or the old vortex balls, and which encloses every b_i such that $d_i \neq 0$.

Then $\deg(u_\varepsilon, \gamma)$ is equal to the sum of the degrees of the old balls enclosed by γ, i.e., N (since $d_i > 0$ and $n = N$), but also to the (algebraic) sum of the degrees of the *new* balls enclosed by γ which is smaller than D. Thus $N \le D$ as claimed. Also note that, since $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \le CN|\log\varepsilon|$, in view of (4.4), D is bounded by CN. Combining this to the relation (4.3), we get

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \omega) \ge \pi N \log \frac{L}{N\varepsilon} - CN.$$

Comparing this to the upper bound coming from the definition of U_N, we find

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \Omega\setminus\omega) \le \pi N^2 \log\frac{1}{L} + O(N^2) \le \pi\eta|\log\varepsilon| + o(|\log\varepsilon|), \tag{11.27}$$

where we have inserted (11.6). This suffices to conclude that $|u| > 1/2$ in $\Omega\setminus\omega$ if η was chosen small enough depending on Ω. Indeed, using Proposition 4.8, the set $\{x \in \Omega \mid |u_\varepsilon(x)| \le 1/2\}$ may be covered by a collection of disjoint closed ball $\mathcal{B}_0$ of total radius $r_0 \le C\varepsilon|\log\varepsilon|$, and such that (using Corollary 3.1) for each $B \in \mathcal{B}_0$,

$$\int_B \frac{\left(1 - |u_\varepsilon|^2\right)^2}{2\varepsilon^2} \ge C,$$

where C is a constant depending on Ω. Then we apply Theorem 5.2 to $(u_\varepsilon, A_\varepsilon)$ with a final radius $r_1 = \varepsilon^{\frac{1}{2}}$. It yields a collection of balls $\mathcal{B}_1$. The radius r_1 is small enough so that for every $B \in \mathcal{B}_1$ we have

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, B) \ge F_\varepsilon(u_\varepsilon, A_\varepsilon, B)\,(1 - o(1)) - O(1), \tag{11.28}$$

but large enough so that (5.27) and (5.28) imply that for every $B \in \mathcal{B}_1$ we have

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B) \ge C|\log\varepsilon|. \tag{11.29}$$

Then (11.27), (11.28) and (11.29) show that if η is small enough, then B_1 contains no ball which is included in $\Omega\setminus\omega$ and therefore $|u| > 1/2$ in $\Omega\setminus\tilde\omega$, where $\tilde\omega$ is the set of $x \in \Omega$ which are at distance less than r_1 from ω. This proves that $\{|u_\varepsilon| < 1/2\}$ is bounded away from $\partial\Omega$. From now on we define ω to be a fixed subdomain of Ω such that $\{|u_\varepsilon| < 1/2\} \subset \omega$, and is bounded away from $\partial\Omega$.

The case $N \to +\infty$

In this case, we assume the Coulomb gauge condition is satisfied. Then we note that $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ is "very locally minimizing" in the sense of Definition 3.8, i.e., for any family $\{x_\varepsilon\}_\varepsilon$ of points in Ω and any $w : \mathbb{R}^2 \to \mathbb{C}$ compactly supported, we have $G_\varepsilon(u_\varepsilon + w_\varepsilon, A_\varepsilon) \geq G_\varepsilon(u_\varepsilon, A_\varepsilon)$, where $w_\varepsilon(x_\varepsilon + \varepsilon y) = w(y)$.

Indeed, since $(u_\varepsilon, A_\varepsilon)$ is a solution of (GL), we have from Corollary 3.1 and Propositions 3.9 and 3.10 that $|\nabla_A u| \leq C/\varepsilon$, $|u| \leq 1$ and $|A| \leq C/\varepsilon$, the last estimate requiring the Coulomb gauge condition. Moreover $\|w_\varepsilon\|_\infty = \|w\|_\infty$ and $\|\nabla w_\varepsilon\|_\infty = \|\nabla w\|_\infty/\varepsilon$. We easily deduce that

$$\left|F_\varepsilon(u_\varepsilon + w_\varepsilon, A'_\varepsilon) - F_\varepsilon(u_\varepsilon, A'_\varepsilon)\right| \leq \frac{C}{\varepsilon^2}\,|\{w_\varepsilon \neq 0\}| \leq C,$$

since $|\{w_\varepsilon \neq 0\}| = \varepsilon^2 |\{w \neq 0\}|$. From (11.25) and since $N \to +\infty$, this implies that for ε small enough, $(u_\varepsilon + w_\varepsilon, A_\varepsilon) \in U_N$ and therefore $G_\varepsilon(u_\varepsilon + w_\varepsilon, A_\varepsilon) \geq G_\varepsilon(u_\varepsilon, A_\varepsilon)$, proving that $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ very locally minimizes G_ε.

$(u_\varepsilon, A_\varepsilon)$ *has exactly* N *zeroes of degree* $+1$. We need to compare more precisely the upper and lower bounds satisfied by $G_\varepsilon(u_\varepsilon, A_\varepsilon)$. Comparing (9.24), (9.40) with the upper bound (11.10) we deduce that

$$\begin{aligned}\int_{\mathcal{B}} |\nabla_{A'_\varepsilon} u_\varepsilon|^2 + \frac{(\operatorname{curl} A'_\varepsilon)^2}{2} + \frac{\left(1 - |u_\varepsilon|^2\right)^2}{2\varepsilon^2} &\leq \pi N \log\frac{L}{\varepsilon} + O(N^2)\\ &= \pi N \log\frac{r}{\varepsilon} + O(N^2).\end{aligned}$$

On the other hand, using the lower bound given by Theorem 4.1 we have

$$\int_{\mathcal{B}'} |\nabla_{A'_\varepsilon} u_\varepsilon|^2 + r'^2\frac{(\operatorname{curl} A'_\varepsilon)^2}{2} + \frac{\left(1 - |u_\varepsilon|^2\right)^2}{2\varepsilon^2} \geq \pi N \log\frac{r'}{N\varepsilon} - CN.$$

Comparing the two and using $N^2 \ll |\log\varepsilon|$ implies that

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon, \mathcal{B}) - F_\varepsilon(u_\varepsilon, A'_\varepsilon, \mathcal{B}') \leq \pi N \log\frac{r(\mathcal{B})}{r(\mathcal{B}')} + o(|\log\varepsilon|).$$

This allows us to apply Proposition 5.2 (it is not difficult to check that its other hypotheses are satisfied) and to deduce that in each ball B_i in

$\mathcal{B}$ included in Ω, u_ε has exactly d_i zeroes of degree $+1$ (recall $d_i > 0$). Since there are no zeroes close to $\partial\Omega$, the claim is proved. Moreover it follows that if we denote by $\{a_k\}$ the zeroes, then

$$\frac{1}{N}\sum_k \delta_{\tilde{a}_k} - \frac{1}{N}\sum_i d_i \delta_{\tilde{b}_i} \to 0, \tag{11.30}$$

where $\tilde{a}_k = (a_k - p)/L$ and $\tilde{b}_k = (b_k - p)/L$. □

Identification of the limit measure. From Theorem 9.1 we have the convergence of a subsequence of $\{\tilde{\mu}(u_\varepsilon, A'_\varepsilon)/N\}_\varepsilon$ to a probability measure μ_* such that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq f_\varepsilon(N) + N^2 I(\mu_*) + o(N^2)$. But from Proposition 9.1 applied to μ_0, the minimizer of I, we can construct a configuration $(v_\varepsilon, B_\varepsilon)$ which belongs to U_N and such that $G_\varepsilon(v_\varepsilon, B_\varepsilon) \leq f_\varepsilon(N) + N^2 I(\mu_0) + o(N^2)$. Using the fact that $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε in U_N, we deduce the same upper bound on $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ hence $I(\mu_*) \leq I(\mu_0)$ and $\mu_* = \mu_0$. Since every subsequence converges to the same limit, $\{\tilde{\mu}(u_\varepsilon, A'_\varepsilon)/N\}_\varepsilon$ converges to μ_0, and $G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(N) + N^2 I(\mu_0) + o(N^2)$.

It remains to check that

$$\frac{1}{N}\sum_k \delta_{\tilde{a}_k} \rightharpoonup \mu_0.$$

From (11.30) this reduces to proving that

$$\frac{1}{N}\tilde{\mu}(u_\varepsilon, A'_\varepsilon) - \frac{1}{N}\sum_i d_i \delta_{\tilde{b}_i} \rightharpoonup 0.$$

We omit the proof since it was done in the course of the proof of Proposition 9.5, where it resulted from (9.68) and (9.67). This finishes the proof of the theorem in the case $N \to +\infty$. □

The case of bounded N

We now assume that N is independent of ε. In this case, the definition of U_N implies that $F_\varepsilon(u_\varepsilon, A'_\varepsilon) \leq C|\log\varepsilon|$, thus we may apply Theorem 5.4 choosing $\eta = \pi/2N$ and the radius

$$r = \min\left(\frac{1}{|\log\varepsilon|}, \frac{1}{h_{\text{ex}}}\right).$$

We find balls $\{B(a_i, R\varepsilon)\}_{1\le i\le K}$ which depend on ε that cover the zero set of u_ε, that are inside Ω, and such that the degree d_i of u_ε on $\partial B(a_i, R\varepsilon)$ is nonzero for every i. Moreover, the lower bound (5.36) holds and from the previous step all the balls are inside ω, since each ball contains at least a point where $u_\varepsilon = 0$.

We have as before

$$N \le \sum_i |d_i|, \tag{11.31}$$

and we use (5.36) together with the definition of U_N to find

$$\pi \sum_i \left({d_i}^2 - \frac{1}{2N} \right) \log \frac{r}{C\varepsilon} \le \pi N \log \frac{L}{\varepsilon} + \pi N^2 \log \frac{1}{L} + O(1).$$

Together with (11.31), this implies that $d_i = 1$ for every i and $\sum_i d_i = N$. We omit the details of the proof which involves a careful study of remainder terms using (11.6) but is not difficult.

Now, since the degrees have been proven to be equal to $+1$ and since the points are bounded away from $\partial\Omega$, we may apply Proposition 10.2, to find

$$F_\varepsilon(u_\varepsilon, A'_\varepsilon) \ge \pi N |\log \varepsilon| - \pi \sum_{i\ne j} \log |a_i - a_j| + \pi \sum_{i,j} S_\Omega(a_i, a_j) + N\gamma + o(1).$$

Plugging this into the energy-splitting lemma (7.22), we have

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge {h_{\mathrm{ex}}}^2 J_0 + \pi N |\log \varepsilon| - \pi \sum_{i\ne j} \log |a_i - a_j| + \pi \sum_{i,j} S_\Omega(a_i, a_j)$$
$$+ h_{\mathrm{ex}} \int_\Omega \mu(u, A')\xi_0 + N\gamma + o(1). \tag{11.32}$$

We claim that

$$h_{\mathrm{ex}} \int_\Omega \mu(u, A')\xi_0 = 2\pi h_{\mathrm{ex}} \sum_i d_i \xi_0(a_i) + o(1). \tag{11.33}$$

Indeed, we know from Theorem 6.1 that this would hold if the a_i's were the centers a'_i of the balls of small radius $r' = C\varepsilon^{\alpha/2}$. But we can easily check (as in the proof of Proposition 9.5 for example) that $2\pi h_{\mathrm{ex}} \sum_i d_i \delta_{a_i} - 2\pi h_{\mathrm{ex}} \sum_i d'_i \delta_{a'_i} \to 0$ as measures, which proves the claim.

Inserting this into (11.32), for $h_{\rm ex} = O(1)$, we deduce that

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq {h_{\rm ex}}^2 J_0 + \pi N |\log\varepsilon| + R_{N,h_{\rm ex}}(a_1, \ldots, a_N) + N\gamma + o(1) \\ &\geq {h_{\rm ex}}^2 J_0 + \pi N |\log\varepsilon| + \min_{\Omega^N} R_{N,h_{\rm ex}} + N\gamma + o(1). \end{aligned}$$

A matching upper bound also holds, with the help of Proposition 10.1, hence we deduce that we have equality and that $(a_1, \ldots, a_N)$ must converge to a minimizer of $R_{N,h_{\rm ex}}$ as $\varepsilon \to 0$.

If $h_{\rm ex} \to +\infty$, then we claim that the a_i's converge to p. The proof is as above: using (7.22) with (11.33) and the lower bound for $F_\varepsilon(u_\varepsilon, A'_\varepsilon)$ coming from the definition of U_N, we have

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq {h_{\rm ex}}^2 J_0 + 2\pi h_{\rm ex} \sum_i d_i \xi_0(a_i) + f_\varepsilon^0(N) - BN^2 + o(1)$$

and comparing it with the upper bound (11.10), we find

$$2\pi h_{\rm ex} \left(\sum_i d_i \xi_0(a_i) - N \underline{\xi_0} \right) \leq CN^2 + o(1).$$

Using $\sum_i d_i = N$ and $h_{\rm ex} \to +\infty$, we deduce $\xi_0(a_i) \to \underline{\xi_0}$ and thus the claim.

It follows that $\pi \sum_{i,j} S_\Omega(a_i, a_j) = \pi N^2 S_\Omega(p,p) + o(1)$. Thus, (11.32) becomes

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq {h_{\rm ex}}^2 J_0 &+ \pi N |\log\varepsilon| - 2\pi N h_{\rm ex} |\underline{\xi_0}| - \pi \sum_{i \neq j} \log|a_i - a_j| \\ &+ 2\pi h_{\rm ex} \sum_i d_i (\xi_0(a_i) - \underline{\xi_0}) + N\gamma + \pi N^2 S_\Omega(p,p) + o(1). \end{aligned}$$

Using Lemma 11.1, this entails

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq {h_{\rm ex}}^2 J_0 + \pi N |\log\varepsilon| - 2\pi N h_{\rm ex} |\underline{\xi_0}| \\ &+ \pi \frac{N^2 - N}{2} \log \frac{h_{\rm ex}}{N} + \min w_N + \pi N^2 S_\Omega(p,p) + N\gamma + o(N^2). \end{aligned}$$

Again with the help of Proposition 10.1, the matching upper bound also holds, hence there is equality above. Thus from Lemma 11.1, the $\tilde{a}_i$'s have to converge to a minimizer of w_N, where $a_i = p + \tilde{a}_i \sqrt{\frac{N}{h_{\rm ex}}}$. This proves assertion 2) and (11.7).

The fact that for ε small enough there are exactly N zeroes of u_ε may be obtained by a blow-up argument around each a_i. Since $|a_i - a_j| \gg \varepsilon$ (they are given by Theorem 5.4) and the a_i's are bounded away from $\partial\Omega$, the blow-ups at the scale ε around each point converge to an entire solution of $-\Delta u = u(1-|u|^2)$ of degree 1 such that

$$\int_{\mathbb{R}^2} \left(1-|u|^2\right)^2 < +\infty,$$

i.e., to the radial degree +1 vortex which has a single zero. Therefore for ε small enough, u_ε has a single zero in $B(a_i, R\varepsilon)$. This finishes the proof of the theorem in the case of N bounded.

Bibliographic notes on Chapter 11: The existence of solutions of the type of Case 1 in the theorem was conjectured by Rubinstein in [158] and first established by Du–Lin in [86]. Their range of existence is here extended.

The main result of existence of branches of solutions for a wide range of h_{ex}, through the local minimization method, in Case 2, was established in [182] in the case of a disc, following [181] which already contained the minimal energy case. Case 3 is new.

Chapter 12

Back to Global Minimization

In this chapter, we establish which solutions, among the ones found in Theorem 11.1, minimize the energy globally. This of course depends on the value $h_{\rm ex}$. As $h_{\rm ex}$ increases, we will see that the minimizers have one, then two, then more and more vortices, as predicted by the physics. This allows us to give precise expansions of the critical fields.

Again we only need to concentrate on the case of $h_{\rm ex}$ close to $H_{c_1}^0$ since, for $h_{\rm ex}$ greater than $H_{c_1}^0$ by at least an order of $|\log\varepsilon|$, the situation is precisely described by Theorem 7.2, while for $h_{\rm ex} - H_{c_1}^0$ much greater than $\log|\log\varepsilon|$, it is described by Theorem 9.2.

12.1 Global Minimizers Close to H_{c_1}

Loosely speaking, from Theorem 11.1 we know that the minimal energy of a solution with n vortices, when n is independent of ε, is equal to $g_\varepsilon(n) + o(1)$, where

$$g_\varepsilon(n) = f_\varepsilon(n) + \min_{(\mathbb{R}^2)^n} w_n + n\gamma. \tag{12.1}$$

Lemma 12.1 (Critical fields). *For every $\varepsilon > 0$, there exists an increasing sequence $\{H_n(\varepsilon)\}_n$, $H_0 = 0$, such that the following holds.*

Given $n \geq 0$ independent of ε, if $h_{ex}(\varepsilon) \gg 1$ is such that

$$g_\varepsilon(n) \leq \min\left(g_\varepsilon(n-1), g_\varepsilon(n+1)\right) + o(1),$$

then

$$H_n - o(1) \leq h_{ex} \leq H_{n+1} + o(1).$$

Moreover, the following asymptotic expansion holds as $\varepsilon \to 0$

$$H_n = \frac{1}{2|\underline{\xi_0}|}\left[|\log\varepsilon| + (n-1)\log\frac{|\log\varepsilon|}{2|\underline{\xi_0}|} + K_n\right] + o(1) \tag{12.2}$$

where

$$K_n = (n-1)\log\frac{1}{n} + \frac{n^2-3n+2}{2}\log\frac{n-1}{n} + \frac{1}{\pi}\left(\min_{(\mathbb{R}^2)^n} w_n - \min_{(\mathbb{R}^2)^{n-1}} w_{n-1} + \gamma + (2n-1)\pi S_\Omega(p,p)\right),$$

γ *was defined in (3.15),* S_Ω *in (7.18),* w_n *in (11.2), and* $\underline{\xi_0}$ *in (7.4)).*

This was illustrated in Fig. 1.7 where the branches of stable solutions with n vortices intersect (i.e., have equal energy) at the H_n's.

Proof. As in the proof of Lemma 9.5, we let $\Delta_n = g_\varepsilon(n) - g_\varepsilon(n-1)$ and we have, using (12.1) and (9.4),

$$\Delta_{n+1} = \pi\left(n\log\frac{h_{\text{ex}}}{n+1} + |\log\varepsilon| + 2h_{\text{ex}}\underline{\xi_0}\right) + R(n+1), \tag{12.3}$$

with

$$R(n+1) = \pi\frac{n^2-n}{2}\log\frac{n}{n+1} + \pi(2n+1)S_\Omega(p,p) + \min_{(\mathbb{R}^2)^{n+1}} w_{n+1} - \min_{(\mathbb{R}^2)^n} w_n + \gamma.$$

As a function of h_{ex}, the function Δ_1 is affine decreasing on $\mathbb{R}_+$ and $\Delta_1(0) > 0$. If $n > 1$, then Δ_n is first increasing and then decreasing. Also, for ε small enough depending on n, if $h_{\text{ex}} = 1$, then Δ_n is strictly positive. Since Δ_n tends to $-\infty$ as $h_{\text{ex}} \to +\infty$, we may again define $H_n(\varepsilon)$ to be the only value of h_{ex} in the interval $[1, +\infty[$ for which Δ_n vanishes. This allows us to define $H_n(\varepsilon)$ for any $n \in \mathbb{N}$ and any $\varepsilon < \varepsilon_0(n)$. The definition of H_n allows us to easily compute the expansion (12.2) from (12.3). It is easy to check that if $1 \ll h_{\text{ex}}$ and n is fixed, then for any $\varepsilon > 0$ small enough we have $\Delta_{n+1} - \Delta_n > 0$. In particular the sequence $\{H_n\}_n$ is increasing.

Now assume $h_{\text{ex}}(\varepsilon) \gg 1$. If

$$g_\varepsilon(n) \le \min\left(g_\varepsilon(n-1), g_\varepsilon(n+1)\right) + o(1),$$

then $\Delta_n \le o(1)$ and $\Delta_{n+1} \ge o(1)$. But the derivative of Δ_n w.r.t $h_{\rm ex}$ is negative and bounded away from 0, thus $H_n - o(1) \le h_{\rm ex} \le H_{n+1} + o(1)$. □

We deduce:

Theorem 12.1. (Global minimizers and critical fields for bounded numbers of vortices). *Assume $N \in \mathbb{N}$. There exists $c_\varepsilon \to 0$ as $\varepsilon \to 0$ such that if $\varepsilon < \varepsilon_0(N)$ and*

$$H_N + c_\varepsilon \le h_{ex} \le H_{N+1} - c_\varepsilon,$$

any global minimizer of G_ε is a solution with N vortices described in Theorem 11.1.

We prove the theorem. Let $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be global minimizers of the energy with $H_N + c_\varepsilon \le h_{\rm ex} \le H_{N+1} - c_\varepsilon$ for some fixed $N \in \mathbb{N}$. Our assumption implies $h_{\rm ex} \le H^0_{c_1} + O(\log|\log\varepsilon|)$ where $H^0_{c_1}$ was defined in (7.16). We wish to prove that for ε small enough we have $(u_\varepsilon, A_\varepsilon) \in U_N$, for a suitable choice of c_ε. However it suffices to prove that $(u_\varepsilon, A_\varepsilon) \in U_n$ for *some* integer n. Indeed, this will prove that it is a minimizer in U_n, hence its energy is $g_\varepsilon(n) + o(1)$. Moreover, by global minimality, this energy will be smaller than the energy of the minimizers in U_{n-1} and U_{n+1} respectively, i.e.,

$$g_\varepsilon(n) \le \min\left(g_\varepsilon(n-1), g_\varepsilon(n+1)\right) + o(1),$$

which by Lemma 12.1 implies $H_n - \delta_\varepsilon \le h_{\rm ex} \le H_{n+1} + \delta_\varepsilon$ for some $\delta_\varepsilon = o(1)$ and then $n = N$, choosing $c_\varepsilon = \delta_\varepsilon/2$.

Note that from $h_{\rm ex} \le C|\log\varepsilon|$, and the fact that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon(1, h_{\rm ex}\nabla^\perp\xi_0) = {h_{\rm ex}}^2 J_0$ we immediately get $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log\varepsilon|^2$ and $F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log\varepsilon|^2$. In particular, as in Chapter 9 we may apply Theorem 4.1 to $(u_\varepsilon, A_\varepsilon)$ to construct, for any $\alpha < \frac{1}{2}$, small balls $\mathcal{B}'$ with total radius $r' = \varepsilon^\alpha$ and grow them using Theorem 4.2 into large balls $\mathcal{B}$ of total radius $r = \frac{1}{\sqrt{h_{\rm ex}}}$. We again denote by $\{d_i\}_i$ the degrees of the large balls and let $n := \sum_i |d_i|$. We now prove that $(u_\varepsilon, A_\varepsilon) \in U_n$, which will conclude the proof of Theorem 12.1.

— *Step 1: $n \ll h_{ex}$.* This is Step 1 of the proof of Theorem 9.2. □

— *Step 2:* $n = O(1)$. Indeed, assume on the contrary that along a subsequence $\{\varepsilon\}$ we have $n \gg 1$. Then, from Theorem 9.1 and Proposition 9.1, and since $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ are global minimizers, we have $G_\varepsilon(u_\varepsilon, A_\varepsilon) = f_\varepsilon(n) + n^2 I(\mu_0) + o(n^2)$, where $f_\varepsilon(n)$ is defined in (9.9). Moreover, by global minimality, this must be less than $g_\varepsilon(N) + o(1)$. But, computing and inserting the inequality $H_N + c_\varepsilon \le h_{\rm ex} \le H_{N+1} - c_\varepsilon$, we find

$$\pi n|\log\varepsilon| - 2\pi n h_{\rm ex}|\underline{\xi_0}| + \frac{\pi}{2}(n^2 - n)\log\frac{h_{\rm ex}}{n} + O(n^2) \\ \le \pi N|\log\varepsilon| - 2\pi N h_{\rm ex}|\underline{\xi_0}| + O(N^2 \log|\log\varepsilon|)$$

hence

$$-CnN\log|\log\varepsilon| + \frac{\pi}{2}(n^2 - n)\log\frac{h_{\rm ex}}{n} \le CN^2\log|\log\varepsilon| + Cn^2.$$

Using $n \to +\infty$ and dividing by $\log\frac{h_{\rm ex}}{n} \to +\infty$, we find

$$n^2 \le \frac{CnN\log|\log\varepsilon|}{\log\frac{h_{\rm ex}}{n}} \le \frac{CnN\log h_{\rm ex}}{\log\frac{h_{\rm ex}}{n}}$$

and writing $\log h_{\rm ex}$ as $\log\frac{h_{\rm ex}}{n} + \log n$, we find $n^2 \le CnN(1 + \log n)$ from which we easily deduce a contradiction with $n \gg 1$. □

From now on, we assume that n is independent of ε and we show that for ε small enough, we have $(u_\varepsilon, A_\varepsilon) \in U_n$.

— *Step 3: The case* $n = 0$. If $n = 0$ then $d_i = 0$ for all i, hence using Proposition 9.3 and since $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le {h_{\rm ex}}^2 J_0$ we deduce that $d_i' = 0$, for every i. Then applying Theorem 6.1 to the small balls, we find that the norm of $\mu(u, A')$ in the dual of $C_0^{0,1}(\Omega)$ is less than $C\varepsilon^p$ for any $p < 1/2$. Using Lemma 7.3 and bounding again $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ by ${h_{\rm ex}}^2 J_0$ we find $F_\varepsilon(u_\varepsilon, A_\varepsilon') \le C\varepsilon^p$ for any $p < 1/2$, and we may bootstrap this information using Theorem 6.1 and Lemma 7.3, taking larger α's, to find $F_\varepsilon(u_\varepsilon, A_\varepsilon') \le C\varepsilon^q$ for any $q \in (0, 1)$, proving that $(u_\varepsilon, A_\varepsilon) \in U_0$ if ε is small enough. □

— *Step 4: The case* $n > 0$. Comparing the minimizer to the solution with n vortices found in Theorem 11.1, we find that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le f_\varepsilon(n) + O(n^2). \tag{12.4}$$

On the other hand, we may apply Proposition 9.4 and (9.40) together with (9.24)–(9.25) yield

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geq f_\varepsilon(n) + O(n^2).$$

Comparing to (12.4) we obtain that the difference between the left- and right-hand sides of (9.40) is $O(1)$ and therefore since $\ell = \sqrt{\frac{n}{h_{\text{ex}}}} \to 0$, $D(t) = n$ for all $t \in [r_0, r_1]$ which implies that $d_i \geq 0$ for all i and $a_i \to p$ for every i such that $d_i \neq 0$. We also obtain from (9.24)–(9.25) that $n = n'$ since $rh_{\text{ex}} = O(\sqrt{|\log \varepsilon|})$ and $\log(r/r') \sim \alpha |\log \varepsilon|$.

Proceeding exactly as in Step 3 of the proof of Proposition 11.3, we deduce that $(u_\varepsilon, A_\varepsilon) \in U_n$ in this case also.

In all cases we have proved $(u_\varepsilon, A_\varepsilon) \in U_n$, which concludes the proof of Theorem 12.1. □

Remark 12.1. *Note that for $N = 0$ and since we have let $H_0 = 0$, the condition above reduces to*

$$h_{ex} \leq \frac{|\log \varepsilon|}{2|\underline{\xi_0}|} + \frac{\gamma + \pi S_\Omega(p,p)}{2\pi |\underline{\xi_0}|} - c_\varepsilon = H_1 - c_\varepsilon.$$

In this case, the global minimizer of G_ε is the Meissner (vortex-free) solution. We have thus shown that if we define the first critical field either as the one below which minimizers are such that $|u|$ does not vanish or as the one above which minimizers have exactly one zero, we have the refined expansion as $\varepsilon \to 0$

$$\boxed{H_{c_1} = H_1 = \frac{|\log \varepsilon|}{2|\underline{\xi_0}|} + \frac{\gamma + \pi S_\Omega(p,p)}{2\pi |\underline{\xi_0}|} + o(1)}.$$

Remark 12.2. *The minimizers of the energy for $h_{ex} \in [H_n, H_{n+1}]$ have exactly n vortices. This does not mean that if one increases the applied field h_{ex} to pass H_{n+1}, an $n+1$-th vortex will really be observed experimentally. Indeed, each configuration with n vortices found in Theorem 11.1 remains a local minimum even for $h_{ex} < H_n$ or $h_{ex} > H_{n+1}$. There is an energy barrier to pass continuously from a configuration with n vortices to a configuration with a different number of vortices. This allows* *for hysteresis phenomena as observed in experiments, i.e., where the system keeps the "memory" of the situation it is coming from, remaining trapped in local minima instead of going to a global minimum.*

The lower and upper fields for which the solutions of Theorem 11.1 lose their linear stability are called respectively the subcooling H_{sc} and superheating H_{sh} fields. Theorem 11.1 shows that $H_{sc} = O(n^2)$ and also that H_{sh} is much larger than H_{c_1}, or than each H_n. In fact, it is expected that $H_{sh} = O(\varepsilon^{-1})$ (it was established for the vortex-free solution in [55], see Chapter 14).

12.2 Possible Generalization: The Case where Λ is not Reduced to a Point

The most general case is that of general domains with Λ not reduced to one point, or $\Lambda = \{p_1, \dots, p_l\}$. Let us still assume that $D^2\xi_0(p_i)$ are definite positive. Looking for solutions with N vortices, we can minimize G_ε again over U_N. It is clear that with the same arguments as used in Proposition 11.3, the minimum is achieved in U_N and yields a locally minimizing solution of (GL), with vortices of degree 1. If $h_{\text{ex}} = O(1)$ and $N = O(1)$, then its vortices converge to a minimizer of $R_{N,h_{\text{ex}}}$, just like in Theorem 11.1. If $N/h_{\text{ex}} \to 0$, then the situation has more structure, and we will show what happens, for example, for the case of N bounded (or fixed) and $h_{\text{ex}} \to +\infty$. As in the proof of Theorem 11.1, we can check that the vortices should all be of degree 1 and tend to Λ. Let us consider that n_1 of them converge to p_1, n_2 to p_2, etc. Let us denote by $a_1, \dots, a_{n_1}$ those converging to p_1, and by $a_{n_1+1}, \dots, a_{n_2}$ those converging to p_2. We may check that writing $\ell = \sqrt{\frac{N}{h_{\text{ex}}}}$ and rescaling by $\tilde{a}_i = \frac{1}{\ell}(a_i - p_k)$ for $n_{k-1} + 1 \le i \le n_k$, if we denote

$$w_k(x_1, \dots, x_{n_k}) = -\pi \sum_{i \neq j \in [1, n_k]} \log|x_i - x_j| + \pi N \sum_{i=1}^{n_k} \langle D^2\xi_0(p_k) x_i, x_i \rangle,$$

we have

$$-\pi\sum_{i\neq j}\log|a_i-a_j|+2\pi h_{\mathrm{ex}}\sum_i\left(\xi_0(a_i)-\underline{\xi_0}\right)$$
$$=\pi\sum_{k=1}^{l}n_k(n_k-1)\log\frac{1}{\ell}-\pi\sum_{i\neq j\in[1,l]}n_in_j\log|p_i-p_j|$$
$$+\sum_{k=1}^{l}w_k\left(\tilde a_{n_1+\cdots+n_{k-1}+1},\dots,\tilde a_{n_1+\cdots+n_k}\right)+o(1). \quad (12.5)$$

We are thus led to minimizing to leading order

$$\sum_{k=1}^{l}n_k(n_k-1)$$

under the constraint $n_1+\cdots+n_l=N$. This is equivalent to finding

$$M_N=\min_{n_1+\cdots+n_l=N}\sum_{k=1}^{l}n_k^2. \quad (12.6)$$

Lemma 12.2. *If $N=ml+r$ with m and r integers, $r<l$, then M_N is achieved for $n_k=m$ in $l-r$ sites and $n_k=m+1$ in r sites. Then*

$$M_N=(m^2-m)l+2mr.$$

Proof. Assume $n_1,\dots,n_l$ is a minimizer. Let $m=\min_{1\le k\le l}n_k$ and $m'=\max_{1\le k\le l}n_k$. Relabelling if necessary, we may assume that $n_1=m$ and $n_2=m'$. Since the configuration is minimizing, it has less energy than that consisting of $n_1+1,n_2-1,n_3,\dots,n_l$. Thus

$$(n_1+1)^2+(n_2-1)^2+\sum_{k\ge 3}n_k^2\ge\sum_{k=1}^{l}n_k^2,$$

that is,

$$2n_1+1-2n_2+1\ge 0$$

or $m-m'+1\ge 0$. Hence we must have $m'\le m+1$. Since m and m' are respectively, the min and the max, we must have $n_k=m$ or $n_k=m+1$ for every k. Let r be the number of k's for which $n_k=m+1$. We have

$r < l$ otherwise $n_k = m+1$ for every k and this would contradict the minimality of m. We have $l-r$ indices such that $n_k = m$, therefore we have

$$(l-r)m + r(m+1) = N$$

or $lm+r = N$. Thus m is indeed the integer part of N/l and the minimizer is as described. □

Inserting this into (12.5) and taking into account all the terms that depend on the p_k's, we are thus led to minimizing

$$\mathcal{W}_N(p_1,\dots,p_l,x_1,\dots,x_N) = -\pi \sum_{i,j\in[1,l]} n_i n_j \log|p_i - p_j| + \sum_{k=1}^{l} w_k\left(x_{n_1+\cdots+n_{k-1}+1},\dots,x_{n_1+\cdots+n_k}\right) + \pi \sum_{k=1}^{l} n_k^2 S_\Omega(p_k,p_k)$$

with the constraint that the $n_1,\dots,n_l$ are minimizers for (12.6). We therefore find an analogue of Theorems 11.1, 12.1:

Theorem 12.2. *Let us assume that $\Lambda = \{p_1,\dots,p_l\}$. Under the same hypotheses as Theorem 11.1, there exists $\varepsilon_0 > 0$ such that for $0<\varepsilon<\varepsilon_0$, there exists $(u_\varepsilon, A_\varepsilon)$ which is a locally minimizing critical point of G_ε, hence a stable solution of (GL), which has exactly N zeroes of degree $a_1^\varepsilon,\dots,a_N^\varepsilon$ of degree one. Moreover,*

1. *If N and h_{ex} are independent of ε, then, possibly after extraction, $(a_1^\varepsilon,\dots,a_N^\varepsilon)$ converges as $\varepsilon\to 0$ to a minimizer of $R_{N,h_{ex}}$.*
2. *If N is independent of ε and $h_{ex}\to+\infty$, then there are n_1 points converging to p_1, n_2 to $p_2,\dots,n_l$ to p_l, and $n_1,\dots,n_l$ minimize (12.6). Moreover, after possible extraction, the configuration of the p_k's and $\tilde{a}_i{}^\varepsilon = (a_i^\varepsilon - p_k)/\ell$ (for $n_{k-1}+1 \le i \le n_k$) converge to a minimizer of $\mathcal{W}_N$ under this constraint, and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = \min_{U_N} G_\varepsilon = {h_{ex}}^2 J_0 + \pi N(|\log\varepsilon| - 2|\underline{\xi_0}|h_{ex}) + \pi(M_N - N)\log\frac{1}{\ell} + \min \mathcal{W}_N + N\gamma + o(1).$$

Next, in order to find the critical fields, one can observe that $M_{n+1} - M_n = 2m$ where again m is the integer part of n/l. Comparing, we find that the solution with n vortices has the least energy between H_n and H_{n+1} with

$$H_n = \frac{1}{2|\underline{\xi_0}|}\left[|\log\varepsilon| + \left(m - \frac{1}{2}\right)\log\frac{|\log\varepsilon|}{2|\underline{\xi_0}|} + K_n\right] + o(1)$$

where

$$K_n = \left(m - \frac{1}{2}\right)\log\frac{1}{n} + (M_{n-1} - n + 1)\log\frac{n-1}{n} \\ + \frac{1}{\pi}\left(\min\mathcal{W}_n - \min\mathcal{W}_{n-1} + \gamma\right),$$

where m is the integer part of $\frac{n-1}{l}$.

It would not be very hard to also generalize Case 3 of Theorem 11.1 to this situation; one would obtain l pockets of vortices centered at each p_i, with $\sim \frac{n}{l}$ vortices in each.

One can also think of generalizing to the case where $D^2\xi_0$ is not positive definite (see the open problems section).

Bibliographic notes on Chapter 12: The existence of the successive critical fields H_n was first established in [181], the asymptotic expansion obtained here is more precise though, since it is up to $o(1)$. The fact that these solutions are global minimizers was established in [169] for h_{ex} below H_{c_1} and in [171] for h_{ex} above H_{c_1}. Finally, the generalization to other domains is new.

Chapter 13

Asymptotics for Solutions

The problem we have dealt with until now was to understand the $\varepsilon \to 0$ limits of the vorticity measures associated to minimizers of the Ginzburg–Landau functional. We now wish to derive a criticality condition for a limiting vorticity measure associated to a family $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ of solutions of the Ginzburg–Landau system (GL) which are not necessarily minimizing.

Intuitively, the force acting on a vortex in the limit $\varepsilon \to 0$ is the gradient of the potential generated by the vortices and the boundary condition. Assume we are in a domain Ω with external magnetic field h_{ex}. We denote by $\{a_i\}_i$ the limiting locations of the vortices and by $\{d_i\}_i$ their degrees. Then the (limiting) potential h is the solution of the London equation

$$\begin{cases} -\Delta h + h = 2\pi \sum_i d_i \delta_{a_i} & \text{in } \Omega \\ h = h_{\text{ex}} & \text{on } \partial\Omega. \end{cases}$$

The criticality condition in this case should be $\nabla h(a_i) = 0$ for every i. Another formulation, letting $\mu = 2\pi \sum_i d_i \delta_{a_i}$, is

$$\mu \nabla h_\mu = 0, \tag{13.1}$$

where h_μ is the solution of

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = h_{\text{ex}} & \text{on } \partial\Omega. \end{cases}$$

This formulation can be extended to vortex densities by considering arbitrary measures μ in (13.1).

Of course the meaning of (13.1) is not clear in many important cases, including that of μ equal to a Dirac mass at p: then h_μ has a logarithmic singularity at p and $\nabla h_\mu(p)$ is not defined. Another example would be when μ is the measure of arclength along a smooth closed curve γ in Ω. Then h_μ is Lipschitz but ∇h_μ is discontinuous on γ, i.e., the support of μ. These types of measures may actually occur as limits of vorticity measures, we will come back to this and give other examples below.

If $h_\mu \in H^1(\Omega)$, or equivalently if the measure μ is in $H^{-1}(\Omega)$ — which is the case in the curve example above — then a weak formulation of (13.1) is at hand. This comes naturally by computing the derivative of $\|h_t\|^2_{H^1(\Omega)}$ with respect to t for variations of h_μ of the form $h_t(x) = h_\mu(x+tX(x))$, where X is a smooth compactly supported vector field in Ω, also called "inner variations". The vanishing of

$$\frac{d}{dt}\|h_t\|^2_{H^1(\Omega)}$$

at $t=0$ for any such X is equivalent to the fact that for $i=1,2$, we have $\partial_1 T_{i1} + \partial_2 T_{i2} = 0$ in the sense of distributions, where

$$T_{ij} = -\partial_i h_\mu \partial_j h_\mu + \frac{1}{2}\left(|\nabla h_\mu|^2 + {h_\mu}^2\right)\delta_{ij}. \tag{13.2}$$

We write, in shorthand, these two equations as $\operatorname{div} T = 0$. This is expressing the fact that h_μ is stationary with respect to inner variations for the functional $\|h\|^2_{H^1(\Omega)}$.

The coefficients T_{ij} are in L^1 if h_μ is in H^1, and therefore $\operatorname{div} T = 0$ makes sense in $\mathcal{D}'(\Omega)$ (the space of distributions). It is a straightforward calculation to check that, if h_μ is smooth enough, C^2 for instance, then

$$\operatorname{div} T = (-\Delta h_\mu + h_\mu)\nabla h_\mu = \mu \nabla h_\mu,$$

hence $\operatorname{div} T = 0$ is the same as (13.1). The relation $\operatorname{div} T = 0$ is thus a weak formulation of (13.1) for nonsmooth fields.

If $h_\mu \notin H^1$, then the tensor T with coefficients T_{ij} need not be in L^1, this is the case if μ is a Dirac mass. In this case we resort to a finite part formulation. More precisely we consider measures μ such that there exists a family $\{E_\delta\}_{\delta>0}$ of sets which become "small" as δ tends to zero and such that $h_\mu, \nabla h_\mu \in L^2(\Omega \setminus E_\delta)$ for every $\delta > 0$. Then the criticality condition for μ will be that for every $\zeta \in \mathcal{D}(\Omega)$ we have, letting

$F_\delta = \zeta^{-1}(\zeta(E_\delta))$,

$$\int_{\Omega\setminus F_\delta} T\nabla\zeta = 0,$$

where $T\nabla\zeta$ is the vector with components $T_{i1}\partial_1\zeta + T_{i2}\partial_2\zeta$. If μ is a Dirac mass at p, we could take $E_\delta = B(p,\delta)$, for instance.

The method to obtain this weak formulation $\operatorname{div} T = 0$ of the limiting condition (13.1) is to pass to the limit in the analogous relation on the stress-energy tensor, denoted here by S_ε (see Definition 3.4)

$$\operatorname{div} S_\varepsilon = 0 \tag{13.3}$$

holding for solutions of (GL), as seen in Proposition 3.7. This relation is a "conservative form" of the Ginzburg–Landau equations, or a corollary of Noether's theorem, again coming naturally from the fact that $(u_\varepsilon, A_\varepsilon)$ is critical for G_ε with respect to inner variations as described above. The task will thus consist in passing to the limit $\varepsilon \to 0$ in the nonlinear relation (13.3).

This problem is very similar to that of passing to the weak limit in solutions of the 2D incompressible Euler equation, if we consider ∇h as the fluid velocity and μ as the fluid vorticity. It is therefore not surprising that the core of our argument is quite similar to that found in DiPerna–Majda [84] in that it uses something analogous to the "concentration-cancellation" property of the weak limits of solutions to 2D Euler. Two specific difficulties we encounter are first the lack of L^1 control of the vorticity, and second the difference between S_ε and (13.2). Note that in our case we are able to prove that "concentration" always occurs whereas it was a hypothesis in [84].

Note that since we have considered an applied field h_{ex} depending on ε, and possibly tending to $+\infty$, and a number of vortices also possibly tending to $+\infty$, there is a normalization issue that we will discuss below. Let us simply mention that if the number of vortices is negligible compared to h_{ex}, then the effect of the boundary is predominant and the criticality condition is simply $\mu\nabla h_0 = 0$, where h_0 is as in (7.1) the solution of $-\Delta h_0 + h_0 = 0$ in Ω and $h_0 = 1$ on $\partial\Omega$. This makes sense for any measure μ since h_0 is smooth.

13.1 Results and Examples

Before stating our results, we need to introduce some definitions.

13.1.1 The Divergence-Free Condition

Definition 13.1 (Divergence-free in finite part). *Assume X is a vector field in Ω. We say X is divergence-free in finite part if there exists a family of sets $\{E_\delta\}_{\delta>0}$ such that*

1. *For any compact $K \subset \Omega$, we have $\lim_{\delta\to 0} \mathrm{cap}_1(K \cap E_\delta) = 0$.*
2. *For every $\delta > 0$, $X \in L^1(\Omega \setminus E_\delta)$.*
3. *For every $\zeta \in C_c^\infty(\Omega)$,*

$$\int_{\Omega\setminus F_\delta} X \cdot \nabla\zeta = 0,$$

where $F_\delta = \zeta^{-1}(\zeta(E_\delta))$.

If T is a 2-tensor with coefficients $\{T_{ij}\}_{1\le i,j\le 2}$, we say that T is divergence-free in finite part if the vectors $T_i = (T_{i1}, T_{i2})$ are, for $i = 1, 2$.

Proposition 13.1. *Assume that X is divergence-free in finite part in Ω and that $X \in L^1(\Omega \setminus E)$. Then for every $\zeta \in C_c^\infty(\Omega)$,*

$$\int_{\Omega\setminus F} X \cdot \nabla\zeta = 0,$$

where $F = \zeta^{-1}(\zeta(E))$. In particular if $X \in L^1(\Omega)$, then $F = \varnothing$ in the above and therefore $\mathrm{div}\, X = 0$ in $\mathcal{D}'(\Omega)$.

Remark 13.1. *A consequence of this proposition is that if X is divergence-free in finite part and is continuous in a neighborhood U of a smooth curve $\gamma = \partial K$, where K is a compact subset of Ω, then $\int_\gamma X \cdot \nu = 0$.*

Indeed, let $\{\zeta_n\}_n$ be a sequence of functions in $C_c^\infty(\Omega)$ converging in $BV(\Omega)$ to $\mathbf{1}_K$, the characteristic function of K, and equal to $\mathbf{1}_K$ outside of U. Then letting $E = \Omega \setminus U$ we have $\Omega \setminus \zeta_n^{-1}(\zeta_n(E)) = \{\zeta_n \neq 0, 1\} \subset U$ and applying Proposition 13.1, we have

$$\int_U X \cdot \nabla\zeta_n = 0.$$

Passing to the limit $n \to +\infty$ proves the desired result.

It will be convenient to use the following:

Definition 13.2. *We say (with some abuse of notation) that a sequence* $\{X_n\}_n$ *in* $L^1(\Omega)$ *converges in* $L^1_\delta(\Omega)$ *to* X *if* $X_n \to X$ *in* $L^1_{loc}(\Omega)$ *except on a set of arbitrarily small* 1*-capacity, or precisely if there exists a family of sets* $\{E_\delta\}_{\delta>0}$ *such that for any compact* $K \subset \Omega$*,*

$$\lim_{\delta\to 0} \mathrm{cap}_1(K \cap E_\delta) = 0, \qquad \forall \delta > 0 \ \lim_{n\to\infty} \int_{K\setminus E_\delta} |X_n - X| = 0. \qquad (13.4)$$

We define similarly the convergence in L^2_δ *by replacing* L^1 *by* L^2 *in the above.*

Note that the limit X need not be in $L^1(\Omega)$.

The rest of this section is devoted to the proof of Proposition 13.1. We recall from Evans–Gariepy [94] that the p-capacity ($1 \le p < 2$) of $E \subset \mathbb{R}^2$ is defined as

$$\mathrm{cap}_p(E) = \inf \left\{ \int_{\mathbb{R}^2} |\nabla\varphi|^p; \varphi \in L^{p^*}(\mathbb{R}^2), \nabla\varphi \in L^p(\mathbb{R}^2), E \subset int\,(\varphi \ge 1) \right\},$$

where $int(A)$ denotes the interior of A and $p^* = 2p/(2-p)$.

Lemma 13.1. *Any bounded set* $A \subset \mathbb{R}^2$ *may be covered by balls* $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ *such that* $\sum_i r_i < C\,\mathrm{cap}_1(A)$*, where* C *is a universal constant. In particular for any Lipschitz function* ζ*,* $\zeta(A)$ *has Lebesgue measure bounded above by* $C\|\zeta\|_{Lip}\,\mathrm{cap}_1(A)$*.*

Proof. This is a restatement of the proof of the property relating cap_1 and $\mathcal{H}^{n-1}$ in [94]. Let $\alpha = \mathrm{cap}_1(A)$ and f be a test function in the definition of α such that

$$\int_{\mathbb{R}^2} |\nabla f| \le 2\alpha.$$

We assume moreover that f is compactly supported. The coarea formula for BV functions (see [94]) applied to f implies that there exists $t \in (1/2, 1)$ such that $E_t = \{f > t\}$ satisfies $\mathrm{per}(E_t) \le 4\alpha$. But A is included in the interior of $\{f \ge 1\}$, hence in the interior of E_t. Therefore for any $x \in A$, the quotient $|B(x,r) \cap E_t|/|B(x,r)|$ is equal to 1 for r small and tends to 0 as $r \to +\infty$. Thus there exists r_x such that it is equal to $1/2$. The relative isoperimetric inequality (see [94]) for sets of finite perimeter then asserts that for any $x \in A$ we have, using the notation of [94],

$$\|\partial E_t\|(B(x, r_x)) \ge C r_x,$$

where $C > 0$ is a universal constant. Extracting a Besicovitch subcovering of A from $\{B(x, r_x)\}_{x\in A}$, and denoting it by $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$, we obtain, by summing the inequalities above,

$$\sum_i r_i \le C \operatorname{per}(E_t) \le C\alpha.$$

The property that $|\zeta(A)| \le C\|\zeta\|_{\mathrm{Lip}} \operatorname{cap}_1(A)$ results by summing the corresponding inequality for each of the balls $B(x_i, r_i)$. □

Proof of Proposition 13.1. Let ζ be a smooth function compactly supported in Ω and $\gamma_t = \{\zeta = t\}$. For any regular value t of ζ, let

$$f(t) = \int_{\gamma_t} T \cdot \nu,$$

where $\nu = \nabla\zeta/|\nabla\zeta|$.

Assuming the hypotheses of Proposition 13.1 are satisfied, T is divergence free in finite part hence there exist sets $\{E_\delta\}_\delta$ satisfying the properties stated in Definition 13.1. We let $F_\delta = \zeta^{-1}(\zeta(E_\delta))$. We begin by proving that for every $\delta > 0$

$$f(t) = 0 \text{ for almost every } t \notin \zeta(E_\delta). \tag{13.5}$$

Indeed, for any smooth $g : \mathbb{R} \to \mathbb{R}$, the coarea formula gives, for any $\delta > 0$ (using $\operatorname{div} T = 0$ in finite part),

$$\int_{\Omega\setminus F_\delta} T \cdot \nabla(g \circ \zeta) = \int_{t\notin\zeta(E_\delta)} g'(t) f(t)\, dt = 0,$$

thus $f(t) = 0$ for a.e. t such that $t \notin \zeta(E_\delta)$.

Using the coarea formula again, we then have, letting $A = \mathbb{R} \setminus \zeta(E)$, and for any $\delta > 0$,

$$\int_{\Omega\setminus F} T \cdot \nabla\zeta = \int_A f(t)\, dt = \int_{A\setminus\zeta(E_\delta)} f(t)\, dt + \int_{A\cap\zeta(E_\delta)} f(t)\, dt.$$

The integral over $A \setminus \zeta(E_\delta)$ is zero from (13.5). Moreover, since $\lim_{\delta\to 0} \operatorname{cap}_1(K \cap E_\delta) = 0$, where K is the support of ζ, and using the previous lemma, the measure of $\zeta(E_\delta)$ goes to zero with δ and thus the integral over $A \cap \zeta(E_\delta)$ also tends to zero as $\delta \to 0$. It follows that $\int_{\Omega\setminus F} T \cdot \nabla\zeta = 0$, proving the proposition. □

13.1.2 Result in the Case with Magnetic Field

We consider $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ a family of solutions of the Ginzburg–Landau equations in Ω. For the sake of generality, we do not impose boundary conditions (this way Ω can be taken to be a subregion of the original domain where the solution is defined), but we assume that $|u_\varepsilon| \le 1$ in Ω and that

$$F_\varepsilon(u_\varepsilon, A_\varepsilon) < C_0 \varepsilon^{\alpha-1}, \quad \alpha > \frac{2}{3} \tag{13.6}$$

for every $\varepsilon > 0$, where α is independent of ε and where F_ε is the free-energy as defined in (4.1). The value 2/3 is a technical limitation. As in the previous chapters, we denote by $\mu_\varepsilon := \mu(u_\varepsilon, A_\varepsilon) = \operatorname{curl}(iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + \operatorname{curl} A_\varepsilon$, $h_\varepsilon = \operatorname{curl} A_\varepsilon$. Also, recall that from Proposition 3.8, any gauge-invariant quantity is smooth in Ω.

In what follows we split the magnetic field h_ε in two pieces: h_ε^0 the field generated by the boundary conditions and h_ε^1 the field generated by the vorticity. More precisely, taking the curl of the second Ginzburg–Landau equation $-\nabla^\perp h_\varepsilon = j_\varepsilon$ we have

$$-\Delta h_\varepsilon + h_\varepsilon = \mu_\varepsilon \text{ in } \Omega. \tag{13.7}$$

Then we define h_ε^0 and h_ε^1 by

$$\begin{cases} -\Delta h_\varepsilon^1 + h_\varepsilon^1 = \mu_\varepsilon & \text{in } \Omega \\ \qquad\qquad h_\varepsilon^1 = 0 & \text{on } \partial\Omega \end{cases}, \quad h_\varepsilon^0 = h_\varepsilon - h_\varepsilon^1. \tag{13.8}$$

Theorem 13.1. (Limiting vorticities for critical points — case with magnetic field).

A) *Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be solutions of the Ginzburg–Landau equations as above. Then for any $\varepsilon > 0$, there exists a measure ν_ε of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ where the sum is finite, $a_i^\varepsilon \in \Omega$ and $d_i^\varepsilon \in \mathbb{Z}$ for every i, such that, letting $n_\varepsilon = \sum_i |d_i^\varepsilon|$,*

$$n_\varepsilon \le C \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_\varepsilon)}{|\log \varepsilon|}, \tag{13.9}$$

where $\mathcal{B}_\varepsilon$ is a union of balls of total radius less than $C\varepsilon^{2/3}$, and such that

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{W^{-1,p}(\Omega)} \|\mu_\varepsilon - \nu_\varepsilon\|_{C^0(\Omega)^*} \to 0, \tag{13.10}$$

for some $p \in (1, 2)$.

B) *Let $\{\nu_\varepsilon\}_\varepsilon$ be any measures of the form $2\pi\sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ satisfying* (13.10), *let $n_\varepsilon = \sum_i |d_i^\varepsilon|$, and let $\{M_\varepsilon\}_\varepsilon$ be positive real numbers such that $\{h_\varepsilon^0/M_\varepsilon\}_\varepsilon$ converges in $L^1_{loc}(\Omega)$ to a function H_0. Then $-\Delta H_0 + H_0 = 0$ in Ω and, possibly after extraction, one of the following holds.*

0. $n_\varepsilon = 0$ for every ε small enough and then μ_ε tends to 0 in $W^{-1,p}(\Omega)$.

1. $n_\varepsilon = o(M_\varepsilon)$ is nonzero for ε small enough, and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ such that

$$\mu\nabla H_0 = 0.$$

hence the support of μ is contained in the set of critical points of H_0.

2. $M_\varepsilon \sim \lambda n_\varepsilon$, with $\lambda > 0$, and then $\mu_\varepsilon/M_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ, and $h_\varepsilon/M_\varepsilon$ converges in $W^{1,p}_{loc}(\Omega)$ to a solution of $-\Delta h_\mu + h_\mu = \mu$ in Ω. Moreover the symmetric 2-tensor T_μ with coefficients T_{ij} given by (13.2) *is divergence-free in finite part in the sense of Definition 13.1.*

3. $M_\varepsilon = o(n_\varepsilon)$, and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ, and $h_\varepsilon/n_\varepsilon$ converges in $W^{1,p}_{loc}(\Omega)$ to the solution of

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu \text{ in } \Omega \\ \qquad\qquad h_\mu = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover, the symmetric 2-tensor T_μ with coefficients T_{ij} given by (13.2) *is divergence-free in finite part.*

In Cases 2) and 3), if $\mu \in H^{-1}(\Omega)$, then T_μ is in $L^1_{loc}(\Omega)$ and divergence-free in the sense of distributions. Moreover $|\nabla h_\mu|^2$ is then in $W^{1,q}_{loc}(\Omega)$ for any $q \in [1,+\infty)$, implying that h_μ is locally Lipschitz. If we assume that $\mu \in L^p(\Omega)$ for some $p > 1$, then

$$\mu\nabla h_\mu = 0 \tag{13.11}$$

almost everywhere in Ω. In Case 3) this implies $\mu = 0$ and $h_\mu = 0$. In Case 2) we get

$$\mu = h_\mu \mathbf{1}_{\{|\nabla h_\mu| = 0\}}. \tag{13.12}$$

Note that since H_0 satisfies $-\Delta H_0 + H_0 = 0$ in Ω, it is necessarily smooth inside Ω.

Application 1 (Natural boundary conditions). *In the case where* $(u_\varepsilon, A_\varepsilon)$ *solves the full system (GL), i.e., the Ginzburg–Landau equations together with the natural boundary conditions* $j_\varepsilon \cdot \nu = 0$ *and* $h_\varepsilon = h_{ex}(\varepsilon)$ *on* $\partial\Omega$*, we may choose* $M_\varepsilon = h_{ex}$ *and then* $h_\varepsilon^0/M_\varepsilon$ *does not depend on* ε*: it is equal to the function* h_0 *which solves* $-\Delta h_0 + h_0 = 0$ *in* Ω *and* $h_0 = 1$ *on* $\partial\Omega$ *(as in (7.1)), and* $H_0 = h_0$*. The results in that case are stated in the introduction, Theorem 1.7. We already noted in Lemma 7.1 that in the case where* Ω *is bounded, smooth and simply connected, this function only has a finite number of critical points, and thus in Case 1), the measure* μ *is a finite linear combination of Dirac masses.*

In Case 2), we may be more precise about h_μ*: dividing* (13.7) *by* h_{ex} *and passing to the limit, we find it solves*

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases}$$

Moreover, assuming that $\mu \in L^p(\Omega)$*, we claim that* $0 \le h_\mu \le 1$*, thus using* (13.12) *we find that* μ *is in fact a nonnegative* L^∞ *function.*

Proof of the claim. To prove that $0 \le h_\mu$, one may multiply (13.12) by $h_- = \min(h_\mu, 0)$ and integrate by parts to obtain

$$\int_\Omega |\nabla h_-|^2 + {h_-}^2 = \int_{\{|\nabla h_\mu|=0\}} {h_-}^2.$$

It follows that h_- is supported in the set where ∇h_μ vanishes, which in turn implies that $h_- = 0$ for if there existed $x_0 \in \Omega$ such that $h_\mu(x_0) < 0$, then considering an arc joining x_0 to $x_1 \in \partial\Omega$, and since $h_\mu(x_1) = 1$, this would imply the existence of some point in the arc where $\nabla h_\mu \neq 0$ and $h_\mu < 0$, a contradiction. Therefore $h_\mu \ge 0$ in Ω.

Similarly, to prove that $h_\mu \le 1$, we let $h_1 = \max(0, h_\mu - 1)$, multiply (13.12) by h_1 and integrate by parts. This yields

$$\int_\Omega |\nabla h_1|^2 + (1 + h_1)h_1 = \int_{\{|\nabla h_\mu|=0\}} (1 + h_1)h_1,$$

implying that h_1 is supported in $\{|\nabla h_\mu| = 0\}$. Arguing by contradiction as above and using the fact that $h_1 = 0$ on $\partial\Omega$, we obtain $h_1 = 0$ in Ω. □

Application 2. (Minimizers of the Ginzburg–Landau functional). *The above result also allows to get a bit more information for example on the minimizers of the Ginzburg–Landau functional described in Chapter 7. Let* $h_{ex} = \lambda|\log \varepsilon|$ *and assume* $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε *in* Ω. *In this case* μ_ε/h_{ex} *converges to a limiting measure uniquely determined by* λ *and denoted* μ_*. *Moreover*

$$\mu_* = \left(1 - \frac{1}{2\lambda}\right) \mathbf{1}_{\omega_\lambda}\, dx,$$

where ω_λ *is a subdomain of* Ω *(see Chapter 7 for these results).*

Let us show what more can be said. Here we assume that we are above the first critical field and $\omega_\lambda \neq \varnothing$.

The argument is as follows: first we construct vortex balls using Theorem 4.1, with total radius $\varepsilon^{1/2}$ *for instance, and let* $\nu_\varepsilon = 2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$. *This is small enough so that* (13.10) *is true and large enough so that* (13.9) *is true (note that* ε^α *would do for any* $\alpha \in (0,1)$*). Then we apply Theorem 13.1 in a subdomain* $\omega \subset \Omega \setminus \omega_\lambda$, *with* $M_\varepsilon = h_{ex}(\varepsilon)$. *In this case* $H_0 = h_{\mu_*}$ *since* $-\Delta h_{\mu_*} + h_{\mu_*} = 0$ *in* ω. *Moreover we claim that* $n_\varepsilon = o(M_\varepsilon)$. *Indeed from Theorem 7.2, the weak limit of the normalized energy densities* $g_\varepsilon(u_\varepsilon, A_\varepsilon)/{h_{ex}}^2$ *in* ω *is the smooth function* $\frac{1}{2}(|\nabla h_{\mu_*}|^2 + {h_{\mu_*}}^2)$. *Therefore, in* (13.9), *the quantity* $F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_\varepsilon)$ *must be* $o({h_{ex}}^2)$ *since* $\mathcal{B}_\varepsilon$ *has measure tending to* 0. *Dividing* (13.9) *by* h_{ex} *then yields* $n_\varepsilon/h_{ex} = o(1)$.

We thus have a better normalization of μ_ε *by restricting to* $\omega \subset \Omega\setminus\omega_\lambda$, *and we then fall into Case 1) of the previous theorem. If we assume* n_ε *to be nonzero for arbitrarily small values of* ε, *we find that* $\mu_\varepsilon/n_\varepsilon$ *tends to a measure* μ *supported in the set of critical points of* h_{μ_*}. *We recall that* h_{μ_*} *solves an obstacle problem (cf. Chapter 7), and if we assume* Ω *to strictly convex for example, we can check that the gradient of* h_{μ_*} *does not vanish outside* ω_λ. *We deduce that* $\mu = 0$ *in* ω.

Recall that μ *is the limit of* $\nu_\varepsilon/n_\varepsilon$. *Thus, assuming for example that every* d_i^ε *is positive, this implies that for every* $\omega \subset \Omega \setminus \omega_\lambda$, *the vortices in* ω *(if there are any) can only accumulate on the boundary of* ω. *This excludes for instance a vortex density outside* ω_λ *that would be small compared to* h_{ex} *but uniform.*

Application 3. *When* μ *is a Dirac mass or a finite linear combination of Dirac masses at* $a_1, \dots, a_n$, *and using the fact that in this case* T_μ *is smooth in* $\Omega \setminus \{a_1, \dots, a_n\}$, *Remark 13.1 implies that if* T_μ *is* *divergence-free in finite part, then, for $r > 0$ small enough, the flux of T_μ on $\partial B(a_i, r)$ is zero. This is precisely the equivalent for the case with magnetic field of the "vanishing gradient property" derived in [43].*

Proof. Indeed, assume μ has a Dirac mass at the origin, of mass 2π for ease of notation. Then, using polar coordinates (r, θ), and letting

$$\nu = \frac{\partial}{\partial r}, \qquad \tau = \frac{1}{r}\frac{\partial}{\partial \theta},$$

we compute $T_\mu \cdot \nu$ in the basis (τ, ν) to find

$$T_\mu \cdot \nu = \frac{1}{2}\left((\partial_\tau h_\mu)^2 - (\partial_\nu h_\mu)^2 + {h_\mu}^2\right)\nu - (\partial_\nu h_\mu \partial_\tau h_\mu)\,\tau.$$

But, we may write $h_\mu = G + H$, where G is the positive solution in $\mathbb{R}^2$ of $-\Delta G + G = 2\pi\delta$ and H is smooth in a neighborhood of 0. Then we have $\partial_\nu G \approx -1/r$ and $\partial_\tau G = 0$ and we get, as $r \to 0$

$$T_\mu \cdot \nu = \frac{1}{2}\left(-\frac{1}{r^2} + 2\frac{\partial_\nu H}{r}\right)\nu + \left(\frac{\partial_\tau H}{r}\right)\tau + O(1).$$

Now we use the fact that the integral $\vec{I}(r)$ of $T_\mu \cdot \nu$ over the circle $\partial B(0, r)$ is zero. Therefore, as $r \to 0$,

$$0 = \nabla H(0) \cdot \vec{I}(r) \approx 2\pi |\nabla H(0)|^2 + o(1),$$

hence $\nabla H(0) = 0$. □

Examples and interpretation

Here we gather examples pertaining to the case of natural boundary conditions. Many examples are provided by minimizers of the Ginzburg–Landau functional in various regimes of the applied field h_{ex}, they all correspond to positive measures. It is an open problem to find solutions with a changing-sign limiting vorticity, if they exist.

We have seen that if $h_{\text{ex}} - H_{c_1} \ll |\log \varepsilon|$ then the limiting measure μ associated to a family of minimizers of the Ginzburg–Landau functional is supported at the minima of h_0, and that in this case n_ε is a $o(h_{\text{ex}})$. This falls into Case 1) of our theorem. Recall that, more generally, this case shows that if the number of vortices is small compared to h_{ex}, then they should all concentrate at the critical points of h_0, which is a finite set of points in Ω, as mentioned in Lemma 7.1. This means that in Case 1), the limiting μ is always a finite linear combination of Dirac masses. This rules out the possibility of nontrivial nonsingular limiting densities of vortices, for example the possibility of lattices of n_ε vortices if $n_\varepsilon \ll h_{\text{ex}}$.

The result of Theorem 7.2 enters in Case 2) and in this case (13.12) is satisfied. Observe that the relation we obtain, weak form of (13.11) can be seen as another rigorous derivation of the mean-field model of Chapman–Rubinstein–Schatzman [72].

Recently Aydi showed in [28] that when Ω is the unit disc, a nonzero vorticity μ which is supported in a finite union of concentric circles can actually arise as limit of the vorticity of some families of solutions. Such measures are in H^{-1} but in this case ∇h_μ is no longer continuous, although $|\nabla h_\mu|^2$ is, thus the strong form (13.11) does not make sense. These examples are constructed by minimizing the Ginzburg–Landau energy among configurations having a well-chosen discrete rotational symmetry.

Further examples where μ is a linear combination of Dirac masses could in principle be of two types: either they would correspond to a number of vortices, as well as an applied field, bounded independently of ε; or to h_{ex} tending to $+\infty$ as $\varepsilon \to 0$ and to a number of vortices of the order of h_{ex}, but concentrating around a finite number of points only. Examples belonging to the first case have been shown to exist in Chapter 11, but it is not known whether the second case can actually occur.

The above examples show that Cases 1) and 2) of the above theorem are not empty, and do not reduce to minimizers of the Ginzburg–Landau functional. It is not known whether Case 3) of the theorem can occur. Against this possibility is the intuition that if the number of vortices is too large compared to the confining field h_{ex}, then they would rather exit Ω. Observe that already we know that μ would have to be singular, because we saw that if $\mu \in L^p$, $p > 1$, in some subdomain of Ω, then $h_\mu = 0$ and then $\mu = 0$ there. However a very symmetric situation may provide an example of such an atypical behavior.

Several more remarks can be made on this theorem.

Remark 13.2. *The definition of n_ε, number of vortices, is not completely natural, or at least not intrinsic. This may be a problem for Case 1, since if one normalizes by a large enough factor, then the limiting measure μ is zero and everything is trivial.*

There are good cases however, for example when $h_{ex}(\varepsilon) = C|\log\varepsilon|$ *and* $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log\varepsilon|^2$*, such as in the case of the obstacle problem in Theorem 7.2. In this case, it is straightforward to check that* $F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C|\log\varepsilon|^2$ *and from* (13.9) *we find*

$$n_\varepsilon \le C|\log\varepsilon|.$$

Thus, in this situation $n_\varepsilon \le Ch_{ex}$*, and in Cases 1 or 2. In Case 2, the normalization is by* h_{ex} *which is more intrinsic.*

For higher values of the energy however, in Case 3, letting $\alpha_\varepsilon = \|h_\varepsilon\|_{H^1}$*, we have, modulo a subsequence, that* $h_\varepsilon/\alpha_\varepsilon$ *converges weakly in* H^1*, hence* $\mu_\varepsilon/\alpha_\varepsilon$ *converges weakly in* H^{-1}*, but it could in principle happen that* $\alpha_\varepsilon = o(n_\varepsilon)$*. In this case, normalizing by* n_ε *would yield zero in the limit, whereas normalizing by* α_ε *could yield a nonzero limit. Such a situation would correspond to solutions with many vortices, the degrees of which are either positive of negative and for the most part cancel out in the limit, leaving however a residual distribution. We do not know if this can actually happen, but if it does, our techniques do not allow us to say anything about the residue, i.e., the limit of* $\mu_\varepsilon/\alpha_\varepsilon$*.*

Remark 13.3. *Concerning the regularity implied by the criticality condition on* μ*, the statements we present do not pretend to be a regularity theory for the equation* $\operatorname{div} T_\mu = 0$*, but rather the direct consequences of the equation, in the spirit of bootstrapping arguments. A variant may be found in [175] where it is proven that* (13.12) *is satisfied assuming* $\nabla h_\mu \in C^0$*,* $|\nabla h_\mu| \in BV$*. A bold conjecture would be that* $\operatorname{div} T_\mu = 0$ *in finite part implies that the support of* μ *is of Hausdorff dimension* 0, 1 *or* 2.

13.1.3 The Case without Magnetic Field

We now consider a family $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions of

$$-\Delta u_\varepsilon = \frac{u_\varepsilon}{\varepsilon^2}(1 - |u_\varepsilon|^2) \quad \text{in } \Omega$$

and again we assume that $|u_\varepsilon| \le 1$ in Ω and

$$E_\varepsilon(u_\varepsilon) < C_0\varepsilon^{\alpha-1}, \quad \alpha > \frac{2}{3} \tag{13.13}$$

for every $\varepsilon > 0$, where E_ε was first defined in (1.2). We let $j_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon)$ and $\mu_\varepsilon = \operatorname{curl} j_\varepsilon$.

Taking the scalar product of the equation with iu_ε yields $(\Delta u_\varepsilon, iu_\varepsilon) = 0$, which by a direct calculation is $\operatorname{div} j_\varepsilon = 0$, hence we may write $j_\varepsilon = \nabla^\perp h_\varepsilon$, where h_ε is the solution of

$$\begin{cases} \Delta h_\varepsilon = \mu_\varepsilon & \text{in } \Omega \\ \partial_\nu h_\varepsilon = j_\varepsilon \cdot \tau & \text{on } \partial\Omega. \end{cases} \tag{13.14}$$

Here and below ν is the outward pointing normal to $\partial\Omega$ and $\tau = \nu^\perp$. By *the solution* to (13.14) or to any other Neumann problem, we will mean the solution with zero average in Ω. As in the case with magnetic field, we split h_ε into two pieces. We define h_ε^0 and h_ε^1 by

$$\begin{cases} -\Delta h_\varepsilon^1 = \mu_\varepsilon & \text{in } \Omega \\ \quad h_\varepsilon^1 = 0 & \text{on } \partial\Omega. \end{cases}, \quad h_\varepsilon^0 = h_\varepsilon - h_\varepsilon^1. \tag{13.15}$$

Theorem 13.2. (Limiting vorticities for critical points — case without magnetic field).

A) *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions of* (1.3) *as above. Then for any $\varepsilon > 0$, there exists a measure ν_ε of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ where the sum is finite, $a_i^\varepsilon \in \Omega$ and $d_i^\varepsilon \in \mathbb{Z}$ for every i, such that, letting $n_\varepsilon = \sum_i |d_i^\varepsilon|$,*

$$n_\varepsilon \le C \frac{E_\varepsilon(u_\varepsilon, \mathcal{B}_\varepsilon)}{|\log \varepsilon|}, \tag{13.16}$$

where $\mathcal{B}_\varepsilon$ is a union of balls of total radius less than $C\varepsilon^{2/3}$, and such that

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{W^{-1,p}(\Omega)} \|\mu_\varepsilon - \nu_\varepsilon\|_{C^0(\Omega)^*} \to 0, \tag{13.17}$$

for some $p \in (1, 2)$.

B) *Let $\{\nu_\varepsilon\}_\varepsilon$ be any measures of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ satisfying* (13.17)*, let $n_\varepsilon = \sum_i |d_i^\varepsilon|$, and let $\{M_\varepsilon\}_\varepsilon$ be positive real numbers such that $\{h_\varepsilon^0 / M_\varepsilon\}_\varepsilon$ converges in $L^1_{loc}(\Omega)$ to a function H_0. Then H_0 is harmonic and, possibly after extraction, one of the following holds.*

0. $n_\varepsilon = 0$ for every ε small enough and then μ_ε tends to 0 in $W^{-1,p}(\Omega)$.

1. *$n_\varepsilon = o(M_\varepsilon)$ is nonzero for ε small enough, and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ such that*

$$\mu\nabla H_0 = 0,$$

hence the support of μ is contained in the set of critical points of H_0.

2. *$M_\varepsilon \sim \lambda n_\varepsilon$, with $\lambda > 0$, and then $\mu_\varepsilon/M_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ, and $h_\varepsilon/M_\varepsilon$ converges in $W^{1,p}_{loc}(\Omega)$ to a solution of $\Delta h_\mu = \mu$ in Ω. Moreover the symmetric 2-tensor T_μ with coefficients T_{ij} given by*

$$T_{ij} = -\partial_i h_\mu \partial_j h_\mu + \frac{1}{2}|\nabla h_\mu|^2 \delta_{ij} \tag{13.18}$$

is divergence-free in finite part.

3. *$M_\varepsilon = o(n_\varepsilon)$, and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ, and $h_\varepsilon/n_\varepsilon$ converges in $W^{1,p}_{loc}(\Omega)$ to the solution of*

$$\begin{cases} \Delta h_\mu = \mu & \text{in } \Omega \\ h_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover the symmetric 2-tensor T_μ with coefficients T_{ij} given by (13.18) *is divergence-free in finite part.*

In Cases 2) or 3), if $\mu \in H^{-1}(\Omega)$, then T_μ is in $L^1_{loc}(\Omega)$ and divergence-free in the sense of distributions and $|\nabla h_\mu|^2$ is smooth, hence h_μ is locally Lipschitz in Ω. If there exists a subdomain Ω' of Ω such that μ is in $L^p(\Omega')$ for some $p > 1$, then $\mu = 0$ in Ω'.

Let us look in detail at the case where u_ε satisfies a Dirichlet boundary condition. Similar results could be proved for Neumann boundary conditions but we prefer Dirichlet for the sake of variety and in order to connect our results to those in [43].

Application 4 (Dirichlet boundary condition). *Assume*

$$|u_\varepsilon| = 1 \text{ on } \partial\Omega \tag{13.19}$$

and that there exist normalizing factors $\{M_\varepsilon\}_{\varepsilon>0}$, and a function $\Phi \in H^{1/2}(\partial\Omega)$ such that

$$\lim_{\varepsilon\to 0} \left\| \frac{j_\varepsilon \cdot \tau}{M_\varepsilon} - \Phi \right\|_{H^{1/2}(\partial\Omega)} = 0. \tag{13.20}$$

From (13.14) *and* (13.15)*, the function* h_ε^0 *is harmonic in* Ω *and such that* $\partial_\nu h_\varepsilon^0 = j_\varepsilon \cdot \tau - \partial_\nu h_\varepsilon^1$ *on* $\partial\Omega$*. In Case 1) we have* $h_\varepsilon^1/M_\varepsilon \to 0$ *therefore* H_0*, which is the limit of* $h_\varepsilon^0/M_\varepsilon$*, solves*

$$\begin{cases} \Delta H_0 = 0 & \textit{in } \Omega \\ \partial_\nu H_0 = \Phi & \textit{on } \partial\Omega. \end{cases}$$

In Case 2), dividing (13.14) *by* M_ε *and passing to the limit, we find that* h_μ *solves*

$$\begin{cases} \Delta h_\mu = \mu & \textit{in } \Omega \\ \partial_\nu h_\mu = \Phi & \textit{on } \partial\Omega. \end{cases}$$

Examples

The case where $M_\varepsilon = 1$ and n_ε is bounded independently of ε, which falls into Case 2) of the above theorem, was treated by Bethuel–Brezis–Hélein in [43] under the stronger hypothesis $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$. In this case, the limiting measure μ is of the type $2\pi\sum_i d_i\delta_{a_i}$, where $d_i \in \mathbb{Z}$ for every i, and it is proved in [43] that the configuration $(a_i, d_i)_i$ is then a critical point of a “renormalized energy” associated to the problem (see [43]), and that this is in turn equivalent to the fact that the function $H(x) = h_\mu(x) + d_i \log|x - a_i|$, which is smooth near a_i, satisfies $\nabla H(a_i) = 0$ (the so-called “vanishing gradient property”). Our function h_μ is denoted by Φ_0 in [43]. We recover this result since as in the case with magnetic field, the vanishing gradient property is equivalent to T_μ being divergence-free in finite part as proved in Application 3.

In the case of a diverging number of vortices, we establish that wherever μ is regular enough (i.e., does not concentrate), then it is 0. This means that, contrary to the case with magnetic field, the Ginzburg–Landau model *without magnetic field* cannot confine a large number of vortices (in particular, cannot capture Abrikosov lattices). An intuitive justification of this fact is provided by the analysis of Sandier–Soret in [176], where it is shown that when the number of points becomes large, the minimizers of the renormalized energy of [43] tend to the boundary of Ω. This can be seen as a double limit $\varepsilon \to 0$, then $n \to +\infty$, whereas Theorem 13.2 treats a simultaneous limit $(\varepsilon, n_\varepsilon) \to (0, +\infty)$, and includes the case of critical points.

The possibility, in this case without magnetic field, of having singular limiting measures, supported on a line for instance, is an open question.

13.2 Preliminary Results

We begin by a result which is a modification of a result whose proof was given to us by A. Ancona [23].

Proposition 13.2. *Assume* $\{\alpha_n\}_n$ *is a sequence of measures such that for some* $p \in (1,2)$

$$\lim_{n\to+\infty} \|\alpha_n\|_{W^{-1,p}(\Omega)} \|\alpha_n\|_{C^0(\Omega)^*} = 0,$$

where $\|\alpha_n\|_{C^0(\Omega)^*}$ *denotes the total variation of* α_n, $\int_\Omega |\alpha_n|$. *Then, letting* h_n *be the solution of*

$$\begin{cases} -\Delta h_n + h_n = \alpha_n & \text{in } \Omega \\ h_n = 0 & \text{on } \partial\Omega, \end{cases}$$

it holds that h_n *and* ∇h_n *converge to* 0 *in* $L^2_\delta(\Omega)$ *(see Definition 13.2).*

Proof. First note that, since $W^{1,q}$ (for $q > 2$) embeds into C^0, the $(C^0)^*$ norm dominates the $W^{-1,p}$ norm for $p < 2$, and thus the hypothesis implies that $\|\alpha_n\|_{W^{-1,p}}$ tends to zero as $n \to +\infty$.

We let

$$\delta_n = \left(\frac{\|\alpha_n\|_{W^{-1,p}}}{\|\alpha_n\|_{C^0(\Omega)^*} + 1} \right)^{1/2}, \quad F_n = \{x \in \Omega \mid |h_n| \geq \delta_n\}. \tag{13.21}$$

Then we have the well-known bound on the p-capacity of F_n (see [94] or [197])

$$\mathrm{cap}_p(F_n) \leq C \frac{\|h_n\|^p_{W^{1,p}}}{{\delta_n}^p}. \tag{13.22}$$

Now we note that by elliptic regularity $\|h_n\|_{W^{1,p}} \leq C\|\alpha_n\|_{W^{-1,p}}$ so from (13.21)–(13.22), we have

$$\mathrm{cap}_p(F_n) \leq C\|\alpha_n\|^{p/2}_{W^{-1,p}} \left(\|\alpha_n\|_{C^0(\Omega)^*} + 1\right)^{p/2},$$

and therefore tends to 0 as $n \to +\infty$. This implies in turn that $\lim_{n\to+\infty} \mathrm{cap}_1(F_n) = 0$.

Also, from a well-known property of Sobolev functions, the truncated function $\bar{h}_n = \max(-\delta_n, \min(h_n, \delta_n))$ satisfies $\nabla \bar{h}_n = 0$ a.e. in F_n, hence

$$\int_{\Omega \setminus F_n} |\nabla h_n|^2 = \int_{\Omega} \nabla h_n \cdot \nabla \bar{h}_n.$$

It follows that

$$\int_{\Omega \setminus F_n} |\nabla h_n|^2 + {h_n}^2 \leq \int_{\Omega} \nabla h_n \cdot \nabla \bar{h}_n + h_n \bar{h}_n = \int_{\Omega} \bar{h}_n \, d\alpha_n,$$

where the last equality follows from $-\Delta h_n + h_n = \alpha_n$. The right-hand side is bounded above by $\delta_n \|\alpha_n\|_{C^0(\Omega)^*}$, hence by $\left(\|\alpha_n\|_{W^{-1,p}} \|\alpha_n\|_{C^0(\Omega)^*}\right)^{1/2}$ and therefore tends to zero as $n \to +\infty$. Thus

$$\lim_{n \to +\infty} \|h_n\|_{L^2(\Omega \setminus F_n)} = \lim_{n \to +\infty} \|\nabla h_n\|_{L^2(\Omega \setminus F_n)} = 0. \tag{13.23}$$

To conclude, since $\lim_{n \to +\infty} \mathrm{cap}_1(F_n) = 0$, there is a subsequence, still denoted by $\{n\}$ such that $\sum_n \mathrm{cap}_1(F_n) < +\infty$. We define

$$E_\delta = \bigcup_{n > \frac{1}{\delta}} F_n.$$

Then $\mathrm{cap}_1(E_\delta)$ tends to zero as $\delta \to 0$ since it is bounded above by the tail of a convergent series. Moreover, for any $\delta > 0$ we have $F_n \subset E_\delta$ when n is large enough and therefore (13.23) implies that $\lim_{n \to +\infty} \|h_n\|_{L^2(\Omega \setminus E_\delta)} = \|\nabla h_n\|_{L^2(\Omega \setminus E_\delta)} = 0$. □

Proposition 13.3. *Assume $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of divergence-free vector fields which converges to X in $L^1_\delta(\Omega)$. Then X is divergence-free in finite part.*

Proof. Consider sets $\{E_\delta\}_\delta$ such that (13.4) is satisfied and assume ζ is a smooth function supported in a compact subset K of Ω. Then, letting $F_\delta = \zeta^{-1}(\zeta(E_\delta))$ and for any n, we have from Proposition 13.1 applied to X_n and E_δ that

$$\int_{\Omega \setminus F_\delta} X_n \cdot \nabla \zeta = 0.$$

Since $E_\delta \subset F_\delta$ and $\{X_n\}_{n\in\mathbb{N}}$ converges to X in $L^1(K \setminus E_\delta)$, we may pass to the limit in the above using the fact that ζ is supported in K to find

$$\int_{\Omega\setminus F_\delta} X \cdot \nabla\zeta = 0.$$

Hence X is divergence-free in finite part. □

Proposition 13.4. *Let $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be solutions of* (GL) *satisfying* (13.6) *and let as usual $h_\varepsilon = \operatorname{curl} A_\varepsilon$. For any $\varepsilon > 0$ we introduce the symmetric 2-tensors T_ε and S_ε whose coefficients are*

$$\begin{aligned} T_{ij} &= -\partial_i h \partial_j h + \frac{1}{2}\left(|\nabla h|^2 + h^2\right)\delta_{ij}, \\ S_{ij} &= (\partial_i^A u, \partial_j^A u) - \frac{1}{2}\left(|\nabla_A u|^2 - h^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2\right)\delta_{ij}, \end{aligned} \tag{13.24}$$

where we have dropped the subscripts ε for T, S, h, u and A for readability.

Then $T_\varepsilon - S_\varepsilon$ tends to 0 in $L^1_\delta(\Omega)$.

Proof. From Proposition 4.8, the set of x in Ω_ε (defined in (4.2)) such that $|u(x)| \le 1 - \varepsilon^{1/3}$ has radius smaller than $C\varepsilon^{\alpha-2/3}$ and therefore there exists a finite union of balls containing this set, with total perimeter bounded by $C\varepsilon^{\alpha-2/3}$. We let Z_ε be this union of balls. Then

$$\lim_{\varepsilon\to 0} \operatorname{cap}_1(Z_\varepsilon) = 0.$$

Indeed, since the 1-capacity of $B(x,r)$ is $2\pi r$ and the capacity is countably subadditive, $\operatorname{cap}_1(Z_\varepsilon) \le C\varepsilon^{\alpha-2/3}$.

The difference between the tensors T_ε and S_ε has a simple expression. We use the notation $u_\varepsilon(x) = \rho(x)e^{i\varphi(x)}$ for u_ε. Now we use the identity $\partial_j u - iA_j u = (\partial_j\rho + i(\partial_j\varphi - A_j))\, e^{i\varphi}$ together with the second Ginzburg–Landau equation $-\nabla^\perp h = \rho^2(\nabla\varphi - A)$ to obtain

$$(\partial_i^A u, \partial_j^A u) = \partial_i\rho\partial_j\rho + \frac{\partial_i^\perp h \partial_j^\perp h}{\rho^2},$$

where we have used the notation $\partial_1^\perp = \partial_2$ and $\partial_2^\perp = -\partial_1$. It follows that

$$\begin{aligned} (\rho^2 S - T)_{ij} = \rho^2 \left(\partial_i\rho\partial_j\rho - \frac{|\nabla\rho|^2}{2}\delta_{ij} - \frac{(1-\rho^2)^2}{4\varepsilon^2}\delta_{ij}\right) \\ + (\rho^2 - 1)\frac{h^2}{2}\delta_{ij}. \end{aligned} \tag{13.25}$$

Let f_ε denote the free-energy density $\frac{1}{2}\left(|\nabla_A u|^2 + h^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)$. The terms on the right-hand side of (13.25) can be bounded by either $C|\nabla\rho|^2$ or $C\varepsilon^{-2}(1-\rho^2)^2$ or, for the last term, by $C(1-\rho^2)f_\varepsilon$. Moreover, the coefficients of $(1-\rho^2)S$ are also clearly bounded by $C(1-\rho^2)f_\varepsilon$. But $1-\rho^2 \le C\varepsilon^{1/3}$ on $\Omega_\varepsilon \setminus Z_\varepsilon$, therefore

$$|T_\varepsilon - S_\varepsilon| \le C\left(\varepsilon^{1/3} f_\varepsilon + |\nabla\rho|^2 + \frac{(1-\rho^2)^2}{2\varepsilon^2}\right) \tag{13.26}$$

holds in $\Omega_\varepsilon \setminus Z_\varepsilon$.

Now let K be a compact subset of Ω. For ε small enough we have $K \subset \Omega_\varepsilon$ therefore (13.26) holds on $K \setminus Z_\varepsilon$: we integrate it on this set. Since $\varepsilon^{1/3}F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C\varepsilon^{\alpha-2/3}$ and $\alpha > 2/3$, the integral of $\varepsilon^{1/3}f_\varepsilon$ tends to zero with ε. We now prove the same for the integral of the remaining terms.

Taking the scalar product of the first Ginzburg–Landau equation with u we obtain

$$-\Delta\rho + \rho|\nabla\varphi - A|^2 = \frac{\rho}{\varepsilon^2}(1-\rho^2).$$

Now we define $\chi : [0,1] \to [0,1]$ as the affine interpolation between the values $\chi(0) = 0$, $\chi(1/2) = 1/2$ and $\chi(1-\varepsilon^{1/3}) = \chi(1) = 1$, then we multiply the above equation by the nonnegative function $\chi(\rho) - \rho$ and integrate in K, that we can assume smooth by enlarging it if necessary. Integrating by parts we obtain

$$\int_K |\nabla\rho|^2 \left(\chi'(\rho) - 1\right) + \rho\left(\chi(\rho) - \rho\right)|\nabla\varphi - A|^2$$

$$= \int_K \rho(1-\rho^2)\frac{\chi(\rho)-\rho}{\varepsilon^2} + \int_{\partial K} \left(\chi(\rho) - \rho\right)\partial_\nu\rho. \tag{13.27}$$

Now, using a mean value argument, we may find a larger compact set K such that $\|\partial_\nu\rho\|^2_{L^2(\partial K)}$ is bounded above by $CF_\varepsilon(u_\varepsilon, A_\varepsilon)$, where C depends on K. Then the boundary term in (13.27) may be bounded above using Cauchy–Schwarz by $CF_\varepsilon(u_\varepsilon, A_\varepsilon)^{1/2}\varepsilon^{1/3}$, hence tends to 0 as $\varepsilon \to 0$. It follows that

$$\int_{K\cap\{\chi(\rho)=1\}} |\nabla\rho|^2 + \frac{\rho(1-\rho^2)(1-\rho)}{\varepsilon^2} + o(1)$$

$$\leq \int_K \frac{\chi(\rho)-\rho}{\rho}\rho^2|\nabla\varphi - A|^2 + \int_{K\cap\{\chi(\rho)\neq 1\}} |\nabla\rho|^2\left(\chi'(\rho)-1\right). \quad (13.28)$$

Now we notice that $\{\chi(\rho)=1\}$ contains the set where $\rho\in(1-\varepsilon^{1/3},1)$ and therefore $\Omega_\varepsilon\setminus Z_\varepsilon$, hence $K\setminus Z_\varepsilon$ if ε is small enough. Moreover when $\rho\geq 1/2$, which is true on the set $\{\chi(\rho)=1\}$, the integrand of the left-hand side is bounded below by $C|\nabla\rho|^2 + C\varepsilon^{-2}(1-\rho^2)^2$ with $C>0$. Therefore the left-hand side of (13.28) bounds from above the integral of the right-hand side of (13.26) on $K\setminus Z_\varepsilon$, for a suitable choice of constant C, and if we want to prove that the latter goes to zero with ε, it suffices to prove the same for the right-hand side of (13.28). For this, we note that $|\chi(\rho)-\rho|/\rho$ is bounded above by $C\varepsilon^{1/3}$ and that where $\chi(\rho)\neq 1$, then $|\chi'(\rho)-1|$ is also bounded above by $C\varepsilon^{1/3}$. It follows that the right-hand side of (13.28) is bounded above by $C\varepsilon^{1/3}F_\varepsilon(u_\varepsilon,A_\varepsilon)$, which is smaller than $C\varepsilon^{\alpha-2/3}$ and therefore tends to zero as ε tends to zero.

We now have defined sets Z_ε such that, as $\varepsilon\to 0$ and for any compact $K\subset\Omega$,

$$\mathrm{cap}_1(Z_\varepsilon)\to 0, \qquad \int_{K\setminus Z_\varepsilon} |T_\varepsilon - S_\varepsilon| \to 0.$$

We choose a decreasing subsequence $\{\varepsilon_n\}$ tending to zero such that $\sum_n \mathrm{cap}_1(Z_{\varepsilon_n})<+\infty$ and let

$$E_\delta = \bigcup_{n>\frac{1}{\delta}} Z_{\varepsilon_n}.$$

Then clearly, $T_{\varepsilon_n}-S_{\varepsilon_n}$ tends to zero in $L^1(K\setminus E_\delta)$ along the subsequence, for any $\delta>0$, and

$$K\cap E_\delta = \bigcup_{n>\frac{1}{\delta}} K\cap Z_{\varepsilon_n},$$

and therefore $\mathrm{cap}_1(K\cap E_\delta)$ tends to zero, since it is bounded above by the tail of a convergent series. The proposition is proved. □

We write as a proposition Part A) of Theorem 13.1.

Proposition 13.5. *Assuming that* $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ *satisfy* (13.6), *there exists for any* $\varepsilon > 0$ *a measure* ν_ε *of the form* $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ *where the sum is finite,* $a_i^\varepsilon \in \Omega$ *and* $d_i^\varepsilon \in \mathbb{Z}$ *for every* i *such that, letting* $n_\varepsilon = \sum_i |d_i^\varepsilon|$,

$$n_\varepsilon \le C \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_\varepsilon)}{|\log \varepsilon|},$$

where $\mathcal{B}_\varepsilon$ *is a union of balls of total radius less than* $C\varepsilon^{2/3}$, *and such that*

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{W^{-1,p}(\Omega)} \|\mu_\varepsilon - \nu_\varepsilon\|_{C^0(\Omega)^*} \to 0, \tag{13.29}$$

for some $p \in (1, 2)$.

Proof. Applying Proposition 4.8 with $\delta = \frac{1}{2}$, there exists a finite collection $\mathcal{B}_\varepsilon^0$ of disjoint closed balls such that $\Omega_\varepsilon \cap \{|u_\varepsilon| < 1/2\} \subset \cup_{B \in \mathcal{B}_\varepsilon^0} B$ and

$$r(\mathcal{B}_\varepsilon^0) \le C\varepsilon^\alpha,$$

where we have used the inequality $F_\varepsilon(|u_\varepsilon|) \le F_\varepsilon(u_\varepsilon, A_\varepsilon) \le C_0 \varepsilon^{\alpha-1}$. Then defining $\mathcal{B}_\varepsilon(t)$ using Theorem 4.2 with $\mathcal{B}_\varepsilon^0$ as the initial collection, we let $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(t_1)$, where t_1 is such that $r(\mathcal{B}_\varepsilon) = \varepsilon^{2/3-\alpha} r(\mathcal{B}_\varepsilon^0) \le C\varepsilon^{2/3}$.

Now we write $\mathcal{B}_\varepsilon = \{B_i^\varepsilon = B(a_i^\varepsilon, r_i^\varepsilon)\}_i$ and let $d_i^\varepsilon = \deg(u_\varepsilon, \partial B_i^\varepsilon)$ if $B_i^\varepsilon \subset \Omega_\varepsilon$ and $d_i^\varepsilon = 0$ otherwise. We then let

$$\nu_\varepsilon = 2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}, \quad n_\varepsilon = \sum_i |d_i^\varepsilon|.$$

From Proposition 4.3 applied to $v_\varepsilon = u_\varepsilon/|u_\varepsilon|$ and using the fact that $|\nabla_A v_\varepsilon|^2 \le 4|\nabla_A u_\varepsilon|^2$ outside the balls of $\mathcal{B}_\varepsilon^0$ we get

$$4F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_\varepsilon) \ge \pi n_\varepsilon \left(\log(\varepsilon^{2/3-\alpha}) - \log 2\right)$$

and dividing by $|\log \varepsilon|$ yields $n_\varepsilon \le C F_\varepsilon(u_\varepsilon, A_\varepsilon, \mathcal{B}_\varepsilon)/|\log \varepsilon|$.

It remains to check (13.29). From Theorem 6.1, and writing $M = F_\varepsilon(u_\varepsilon, A_\varepsilon)$, we have

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{(C_0^{0,1}(\Omega))^*} \le C\varepsilon^{2/3} M, \quad \|\mu_\varepsilon - \nu_\varepsilon\|_{(C^0)^*} \le CM,$$

the second inequality coming from the bound $\|\mu_\varepsilon\|_{(C^0)^*} \le CM$ of Theorem 6.1 added to the bound $\|\nu_\varepsilon\|_{(C^0)^*} = n_\varepsilon \le CM/|\log \varepsilon|$. Then interpolating by Lemma 6.5 and using $r(\mathcal{B}_\varepsilon^0) \le C\varepsilon^{2/3}$ we find for any $\beta \in (0, 1)$,

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{(C_0^{0,\beta}(\Omega))^*} \|\mu_\varepsilon - \nu_\varepsilon\|_{(C^0)^*} \le CM^2 \varepsilon^{2\beta/3}.$$

Since $M < C\varepsilon^{\alpha-1}$ for some $\alpha > 2/3$, it follows that, choosing β close enough to 1, the right-hand side is bounded above by a positive power of ε and thus tends to 0 as $\varepsilon \to 0$.

We conclude by recalling that there exists $q > 2$ such that $W_0^{1,q}$ embeds into $C_0^{0,\beta}$ and therefore by duality $(C_0^{0,\beta})^*$ embeds into $W^{-1,p}$, for some $p \in (1,2)$. For such a p, (13.29) is satisfied. $\square$

13.3 Proof of Theorem 13.1, Criticality Conditions

Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be solutions of the Ginzburg–Landau equations satisfying (13.6) and as usual let $h_\varepsilon = \operatorname{curl} A_\varepsilon$, $j_\varepsilon = (iu_\varepsilon, \nabla_A u_\varepsilon)$ and $\mu_\varepsilon = \operatorname{curl} j_\varepsilon + h_\varepsilon$.

If $n_\varepsilon = 0$ for every small enough ε, then (13.10) implies that the $W^{-1,p}$ norm of μ_ε goes to zero since $\nu_\varepsilon = 0$ in this case and moreover, the $(C^0)^*$ norm is stronger than the $W^{-1,p}$ norm.

Otherwise, $(\mu_\varepsilon - \nu_\varepsilon)/n_\varepsilon$ tends to zero in $W^{-1,p}$ for some $p \in (1,2)$ while by definition $\nu_\varepsilon/n_\varepsilon$ is a bounded sequence of measures. Therefore a subsequence converges in the weak sense of measures. On the other hand, by Ascoli's theorem, for any $\alpha > 0$, $C^{0,\alpha}(\Omega)$ embeds compactly into $C^0(\Omega)$, thus $(C^0)^*$ embeds compactly into $(C^{0,\alpha})^*$ and into $W^{-1,p}$ for $p < 2$ (by embedding of $W^{1,q}$ into $C^{0,\alpha}$ for $q > 2$ and appropriate α). We may thus assume, after extraction of a subsequence, that $\nu_\varepsilon/n_\varepsilon$ converges strongly in $W^{-1,p}(\Omega)$ for $p \in (1,2)$, and then $\mu_\varepsilon/n_\varepsilon$ converges to a *measure*, strongly in $W^{-1,p}$, for some $p \in (1,2)$. If $M_\varepsilon \sim \lambda n_\varepsilon$ with $\lambda > 0$, then the same is true for $\mu_\varepsilon/M_\varepsilon$.

Assume $M_\varepsilon \sim \lambda n_\varepsilon$ with $\lambda > 0$. From (13.8), and since $\mu_\varepsilon/M_\varepsilon$ converges to μ in $W^{-1,p}$, we deduce that $h_\varepsilon^1/M_\varepsilon$ converges in $W^{1,p}$ to the solution of $-\Delta h_1 + h_1 = \mu$ in Ω and $h_1 = 0$ on $\partial\Omega$. On the other hand $h_\varepsilon^0/M_\varepsilon$, which is a solution to $\Delta f = f$ from (13.8) and converges in $L^1_{loc}(\Omega)$ to H_0, in fact, converges in $C^k_{\text{loc}}(\Omega)$ for any k. Therefore $h_\varepsilon/M_\varepsilon$ converges in $W^{1,p}_{\text{loc}}(\Omega)$ to $h_\mu = H_0 + h_1$, which satisfies $-\Delta h_\mu + h_\mu = \mu$ in Ω.

If $M_\varepsilon = o(n_\varepsilon)$, then $h_\varepsilon^0/n_\varepsilon$ tends to 0 in $C^k_{\text{loc}}(\Omega)$ for any k, while as above $h_\varepsilon^1/n_\varepsilon$ converges in $W^{1,p}$ to the solution of $-\Delta h_\mu + h_\mu = \mu$ in Ω and $h_\mu = 0$ on $\partial\Omega$, where μ is the limit of $\mu_\varepsilon/n_\varepsilon$. Thus $h_\varepsilon/n_\varepsilon$ converges in $W^{1,p}_{\text{loc}}(\Omega)$ to h_μ.

We now derive the criticality conditions satisfied by μ.

Proof of Theorem 13.1, Items 2 and 3.

Since $(u_\varepsilon, A_\varepsilon)$ is a solution of the Ginzburg–Landau equations and from Proposition 3.4, the symmetric 2-tensor S_ε with coefficients defined by (13.24) is divergence-free. Moreover, from Proposition 13.4, we get that

$$T_\varepsilon - S_\varepsilon \text{ converges to } 0 \text{ in } L^1_\delta(\Omega). \tag{13.30}$$

where T_ε is the symmetric 2-tensor with coefficients

$$T_{ij} = -\partial_i h_\varepsilon \partial_j h_\varepsilon + 1/2 \left(|\nabla h_\varepsilon|^2 + {h_\varepsilon}^2\right) \delta_{ij}.$$

From (13.8), we have the decomposition $h_\varepsilon = h^0_\varepsilon + h^1_\varepsilon$. We further decompose h^1_ε as $U_\varepsilon + V_\varepsilon$, where

$$\begin{cases} -\Delta U_\varepsilon + U_\varepsilon = \mu_\varepsilon - \nu_\varepsilon & \text{in } \Omega \\ U_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta V_\varepsilon + V_\varepsilon = \nu_\varepsilon & \text{in } \Omega \\ V_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

In Case 2) of the Theorem, we have $M_\varepsilon \sim \lambda n_\varepsilon$ with $\lambda > 0$ and μ is the limit of either $\mu_\varepsilon/M_\varepsilon$ or $\nu_\varepsilon/M_\varepsilon$. From (13.10) we may apply Proposition 13.2 to $\alpha_\varepsilon = (\mu_\varepsilon - \nu_\varepsilon)/M_\varepsilon$ and find that

$$\frac{U_\varepsilon}{M_\varepsilon} \text{ and } \nabla\left(\frac{U_\varepsilon}{M_\varepsilon}\right) \text{ tend to } 0 \text{ in } L^2_\delta(\Omega). \tag{13.31}$$

Also, since $\{\nu_\varepsilon/M_\varepsilon\}_\varepsilon$ is a bounded sequence of measures, it converges in $C^0_0(\Omega)^*$ hence in $W^{-1,p}$ and we may apply Proposition 13.2 to $\beta_\varepsilon = (\nu_\varepsilon/M_\varepsilon) - \mu$ to find that

$$\frac{V_\varepsilon}{M_\varepsilon} - h_1 \text{ and } \nabla\left(\frac{V_\varepsilon}{M_\varepsilon} - h_1\right) \text{ tend to } 0 \text{ in } L^2_\delta(\Omega), \tag{13.32}$$

where h_1 is the limit of $h^1_\varepsilon/M_\varepsilon$.

In view of (13.31)–(13.32) and since $\{h^0_\varepsilon/M_\varepsilon\}_\varepsilon$ converges in $C^k_{\text{loc}}(\Omega)$ to H_0, hence in $L^2_\delta(\Omega)$ also, we deduce that $h_\varepsilon/M_\varepsilon$ and its gradient both converge to $h_\mu = H_0 + h_1$ in $L^2_\delta(\Omega)$. In particular, defining T_μ as the tensor with coefficients $T^\mu_{ij} = -\partial_i h_\mu \partial_j h_\mu + 1/2\left(|\nabla h_\mu|^2 + {h_\mu}^2\right)\delta_{ij}$, we get that

$$\frac{T_\varepsilon}{{M_\varepsilon}^2} \text{ converges to } T_\mu \text{ in } L^1_\delta(\Omega). \tag{13.33}$$

In Case 3) of the theorem, where $M_\varepsilon = o(n_\varepsilon)$, μ is the limit of $\mu_\varepsilon/n_\varepsilon$ or $\nu_\varepsilon/n_\varepsilon$, and we proceed as above, normalizing by n_ε instead of M_ε to find that

$$\frac{T_\varepsilon}{{n_\varepsilon}^2} \text{ converges to } T_\mu \text{ in } L^1_\delta(\Omega). \tag{13.34}$$

In Case 2) (resp. case 3)), from (13.30), (13.33) (resp. (13.34)) we find that $S_\varepsilon/({M_\varepsilon}^2)$ (resp. $S_\varepsilon/({n_\varepsilon}^2)$) converges to T_μ in $L^1_\delta(\Omega)$. Moreover, Proposition 13.3 allows us to conclude that since S_ε is divergence-free, the tensor T_μ is divergence-free in finite part.

Proof of Theorem 13.1, Item 1.

We use again the decomposition

$$h_\varepsilon = h^0_\varepsilon + h^1_\varepsilon, \quad h^1_\varepsilon = U_\varepsilon + V_\varepsilon.$$

We have a corresponding decomposition for the tensor T_ε

$$T_\varepsilon = T^{00}_\varepsilon + T^{01}_\varepsilon + T^{10}_\varepsilon + T^{11}_\varepsilon, \tag{13.35}$$

where T^{ab}_ε denotes the tensor with coefficients

$$T^{ab}_{ij} = -\partial_i h^a_\varepsilon \partial_j h^b_\varepsilon + \frac{1}{2}\left(\nabla h^a_\varepsilon \cdot \nabla h^b_\varepsilon + h^a_\varepsilon h^b_\varepsilon\right)\delta_{ij}.$$

From (13.31)–(13.32) above, we deduce that $h^1_\varepsilon/n_\varepsilon$, $\nabla h^1_\varepsilon/n_\varepsilon$ converge to h_μ, ∇h_μ in $L^2_\delta(\Omega)$, where $-\Delta h_\mu + h_\mu = \mu$ in Ω and $h_\mu = 0$ on $\partial\Omega$. Since $n_\varepsilon = o(M_\varepsilon)$, this implies that

$$\frac{h^1_\varepsilon}{\sqrt{n_\varepsilon M_\varepsilon}}, \quad \nabla\left(\frac{h^1_\varepsilon}{\sqrt{n_\varepsilon M_\varepsilon}}\right)$$

both converge to 0 in $L^2_\delta(\Omega)$ and then that

$$\frac{T^{11}_\varepsilon}{n_\varepsilon M_\varepsilon} \text{ converges to } 0 \text{ in } L^1_\delta(\Omega). \tag{13.36}$$

On the other hand, we know that $h_\varepsilon^0/M_\varepsilon$ converges in $C^k_{\text{loc}}(\Omega)$ to H_0 and that $h_\varepsilon^1/n_\varepsilon$ converges to h_μ in $W^{1,p}(\Omega)$. We deduce that

$$\frac{T_\varepsilon^{01}+T_\varepsilon^{10}}{n_\varepsilon M_\varepsilon} \text{ converges to } T'_\mu \text{ in } L^1_{\text{loc}}(\Omega), \tag{13.37}$$

where T'_μ is the tensor with coefficients

$$(T'_\mu)_{ij} = -\partial_i H_0 \partial_j h_\mu - \partial_j H_0 \partial_i h_\mu + (\nabla H_0 \cdot \nabla h_\mu + H_0 h_\mu)\,\delta_{ij}.$$

It follows from (13.35), (13.36), (13.37) that $(T_\varepsilon - T_\varepsilon^{00})/(n_\varepsilon M_\varepsilon)$ converges to T'_μ in $L^1_\delta(\Omega)$ and then using Proposition 13.4 that $(S_\varepsilon - T_\varepsilon^{00})/(n_\varepsilon M_\varepsilon)$ converges to T'_μ in $L^1_\delta(\Omega)$. But we know that S_ε is divergence-free and, using the fact that $-\Delta h_\varepsilon^0 + h_\varepsilon^0 = 0$ in Ω and in particular, smooth, it is straightforward to compute $\operatorname{div} T_\varepsilon^{00} = 0$. It follows that T'_μ is divergence-free in finite part, hence divergence-free in the sense of distributions since it belongs to $L^1_{\text{loc}}(\Omega)$ (see Proposition 13.1).

Now since H_0 is smooth in Ω, the Leibnitz rule may be used to compute the distributional divergence of T'_μ and it is easy to check that $\operatorname{div} T'_\mu = \mu \nabla h_0$, hence we have established $\mu \nabla h_0 = 0$.

13.4 Proof of Theorem 13.1, Regularity Issues

We now proceed to proving the remaining assertions of Theorem 13.1, which describe some consequences of the fact that T_μ is divergence-free in finite part in special cases.

Properties Assuming $\mu \in H^{-1}$

In this case, the limit of $h_\varepsilon^1/n_\varepsilon$ belongs to $H^1(\Omega)$ while H_0 is smooth inside Ω. Therefore in Case 2), the limit h_μ of $h_\varepsilon/M_\varepsilon$ is in $H^1_{\text{loc}}(\Omega)$ while in Case 3), h_μ is in $H^1(\Omega)$. In any case, $T_\mu \in L^1_{\text{loc}}(\Omega)$ and Proposition 13.1 tells us that $\operatorname{div} T_\mu = 0$ in the sense of distributions in Ω.

Assuming we are in Case 2), we let

$$X = \frac{1}{2}\left((\partial_2 h_\mu)^2 - (\partial_1 h_\mu)^2, -2\partial_1 h_\mu \partial_2 h_\mu\right).$$

Then $X = (T_{11} - {h_\mu}^2/2, T_{12})$ and $X = (-T_{22} - {h_\mu}^2/2, T_{21})$, where the T_{ij}'s are the coefficients of T_μ. It follows from $\operatorname{div} T_\mu = 0$ that $\operatorname{div} X = -h_\mu \partial_1 h_\mu$, $\operatorname{curl} X = h_\mu \partial_2 h_\mu$.

Let now f_1 be a solution of $\Delta f_1 = -h_\mu \partial_1 h_\mu$ in Ω and f_2 a solution of $\Delta f_2 = -h_\mu \partial_2 h_\mu$. Since $h_\mu \in H^1_{\text{loc}}(\Omega)$, by Sobolev embedding we have $h_\mu \nabla h_\mu \in L^p_{\text{loc}}(\Omega)$ for any $p \in [1,2)$ and therefore f_1 and f_2 are in $W^{2,p}_{\text{loc}}(\Omega)$ for any $p \in [1,2)$, and thus in $W^{1,q}_{\text{loc}}(\Omega)$ for any $q \in [1,+\infty)$.

Then, since $\Delta f_1 = \operatorname{div} X$ and $\Delta f_2 = \operatorname{curl} X$, we have $X = \nabla f_1 + \nabla^\perp f_2 + Y$, where Y satisfies $\operatorname{div} Y = \operatorname{curl} Y = 0$ in $\mathcal{D}'(\Omega)$. Thus Y is a harmonic, hence smooth, vector field in Ω. It follows that $X \in L^q_{\text{loc}}(\Omega)$ for any $q \in [1,+\infty)$. On the other hand, a direct calculation yields $4|X|^2 = |\nabla h_\mu|^4$, hence we get $|\nabla h_\mu| \in L^q_{\text{loc}}(\Omega)$, for any $q \in [1,+\infty)$. Bootstrapping the argument, we find $h_\mu \nabla h_\mu \in L^q_{\text{loc}}$ for any q, then $f_1, f_2 \in W^{2,q}_{\text{loc}}$, and $X \in W^{1,q}_{\text{loc}}$ and finally $|\nabla h_\mu|^2 = 2|X| \in W^{1,q}_{\text{loc}}$ for any $q \in [1,+\infty)$. By Sobolev embedding, this implies that $|\nabla h_\mu|$ is bounded locally in Ω, hence h_μ is locally Lipschitz in Ω.

The case where $M_\varepsilon = o(n_\varepsilon)$ is identical.

Properties Assuming $\mu \in L^p(\Omega)$, $p > 1$

Note that this is a subcase of the previous one. Indeed, the embedding of H^1 into any L^q, $q < +\infty$ implies the embedding of any L^p, $p > 1$ into H^{-1}. Thus in this case the previous section implies that ∇h_μ is in $L^\infty_{\text{loc}}(\Omega)$.

In Cases 2) or 3) of the Theorem, we define a sequence $\mu_n = \mu * \rho_n$ obtained by convolution of μ with a regularizing kernel $\{\rho_n\}_n$. We define $h_n = h_\mu * \rho_n$ and let T_n be the tensor with coefficients $-\partial_i h_n \partial_j h_n + 1/2\left(|\nabla h_n|^2 + {h_n}^2\right)\delta_{ij}$. Then μ_n tends to μ in L^p and, since $\nabla h_\mu \in L^\infty_{\text{loc}}$, ∇h_n tends to ∇h_μ in $L^q_{\text{loc}}(\Omega)$, for any $q \in [1,+\infty)$, implying that

$$\mu_n \nabla h_n \to \mu \nabla h_\mu, \quad T_n \to T_\mu$$

in $L^1_{\text{loc}}(\Omega)$.

It follows that $\operatorname{div} T_n \to \operatorname{div} T_\mu = 0$ and that $\mu_n \nabla h_n \to \mu \nabla h_\mu$ in $\mathcal{D}'(\Omega)$. Moreover $\operatorname{div} T_n = (-\Delta h_n + h_n)\nabla h_n$, from usual calculus, and $-\Delta h_n + h_n = \mu_n$ by the properties of convolution, hence $\operatorname{div} T_n = \mu_n \nabla h_n$. Passing to the limit, we get $\mu \nabla h_\mu = \lim_n \operatorname{div} T_n = 0$ in $L^1_{\text{loc}}(\Omega)$, hence a.e.

Now, from a well-known property of Sobolev functions we have $\Delta h_\mu = 0$ a.e. on the set $F = \{\nabla h_\mu = 0\}$. Thus $\mu = h_\mu$ a.e. on the set F, while $\mu = 0$ a.e. on the complement of F from the identity $\mu \nabla h_\mu = 0$. We conclude that

$$\mu = h_\mu \mathbf{1}_{\{|\nabla h_\mu| = 0\}}.$$

In the case $M_\varepsilon = o(n_\varepsilon)$, multiplying this equation by h_μ and integrating by parts the left-hand side, we find $h_\mu = 0$ in Ω, and thus $\mu = 0$.

13.5 The Case without Magnetic Field

The proof of Theorem 13.2 follows very closely that of Theorem 13.1. We will therefore leave some of the details to the reader. We begin by very close versions of the Propositions 13.2, 13.4 which we do not prove. Note that Proposition 13.3 and 13.5 may be used as such in the case without magnetic field.

Proposition 13.6. *Assume $\{\alpha_n\}_n$ is a sequence of measures such that for some $p \in (1,2)$*

$$\lim_{n\to+\infty} \|\alpha_n\|_{W^{-1,p}(\Omega)} \|\alpha_n\|_{C^0(\Omega)^*} = 0,$$

where $\|\alpha_n\|_{C^0(\Omega)^*}$ denotes the total variation of α_n, $\int_\Omega |\alpha_n|$. Then, letting h_n be the solution of

$$\begin{cases} -\Delta h_n = \alpha_n & \text{in } \Omega \\ \quad h_n = 0 & \text{on } \partial\Omega, \end{cases}$$

it holds that h_n and ∇h_n converge to 0 in $L^2_\delta(\Omega)$.

Proposition 13.7. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions of $-\Delta u_\varepsilon = \varepsilon^{-2} u_\varepsilon(1-|u_\varepsilon|^2)$ satisfying (13.13), (13.19) and (13.20). We define h_ε by (13.14) and for any $\varepsilon > 0$ we define the symmetric 2-tensors T_ε, S_ε with coefficients*

$$\begin{aligned} T_{ij} &= -\partial_i h \partial_j h + \frac{1}{2}|\nabla h|^2 \delta_{ij}, \\ S_{ij} &= (\partial_i u, \partial_j u) - \frac{1}{2}\left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2\right)\delta_{ij}, \end{aligned} \tag{13.38}$$

where we have dropped the subscripts ε for T, S, h and u for readability.

Then $T_\varepsilon - S_\varepsilon$ tends to 0 in $L^1_\delta(\Omega)$.

Criticality Conditions

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions of $-\Delta u_\varepsilon = \varepsilon^{-2} u_\varepsilon(1-|u_\varepsilon|^2)$. We let $j_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon)$ and $\mu_\varepsilon = \operatorname{curl} j_\varepsilon$. The function h_ε is defined by (13.14). We assume the energy bound (13.13), and the boundary conditions (13.19) and (13.20) are satisfied.

Proposition 13.5 may be applied to $(u_\varepsilon, A_\varepsilon = 0)$ to yield measures $\{\nu_\varepsilon\}_\varepsilon$ satisfying (13.16), (13.17).

If $n_\varepsilon = 0$ for ε small, we find as in the case with magnetic field that μ_ε converges to 0 in $W^{-1,p}$, for some $p \in (1,2)$. Otherwise, exactly as in the case with magnetic field, we find that, modulo a subsequence, $\mu_\varepsilon/n_\varepsilon$ converges to a *measure* in $W^{-1,p}$, for some $p \in (1,2)$. In Case 2) the same is true of $\mu_\varepsilon/M_\varepsilon$. We now derive the criticality conditions satisfied by the limiting measure μ.

Proof of Theorem 13.2, Items 2 and 3. We already saw in (5.6)–(5.7) that, for solutions of $-\Delta u_\varepsilon = \varepsilon^{-2} u_\varepsilon(1 - |u_\varepsilon|^2)$, the symmetric 2-tensor S_ε with coefficients defined by (13.38) is divergence-free. As in the case with magnetic field, Proposition 13.7 implies the existence of a subsequence that we still denote by $\{\varepsilon\}$ such that $S_\varepsilon - T_\varepsilon$ tends to 0 in $L^1_\delta(\Omega)$, where T_ε is the symmetric 2-tensor with coefficients $T_{ij} = -\partial_i h_\varepsilon \partial_j h_\varepsilon + 1/2|\nabla h_\varepsilon|^2 \delta_{ij}$.

We use the decomposition (13.15), and decompose h^1_ε again as $U_\varepsilon + V_\varepsilon$ as in the case with magnetic field, replacing the operator $-\Delta + 1$ by Δ. Then, in Case 2), using (13.17) and Proposition 13.6, we find that (13.31) and (13.32) are true, implying (13.33), where T_μ now denotes the tensor with coefficients $-\partial_i h_\mu \partial_j h_\mu + |\nabla h_\mu|^2 \delta_{ij}$. Similarly we obtain (13.34) in Case 1). Together with the fact that $S_\varepsilon - T_\varepsilon$ tends to 0 in $L^1_\delta(\Omega)$, we then obtain that T_μ is divergence-free in finite part in both cases. □

Proof of Theorem 13.2, Item 1. We now assume $n_\varepsilon = o(M_\varepsilon)$. We keep the same notation as above, and decompose T_ε again as in (13.35), defining the T^{ab} as the tensor with coefficients

$$T^{ab}_{ij} = -\partial_i h^a_\varepsilon \partial_j h^b_\varepsilon + \frac{1}{2} \nabla h^a_\varepsilon \cdot \nabla h^b_\varepsilon \delta_{ij}.$$

The rest of the argument follows as in the case with magnetic field, and proves that

$$\frac{T^{01}_\varepsilon + T^{10}_\varepsilon}{n_\varepsilon M_\varepsilon}$$

converges to T'_μ in $L^1_{\mathrm{loc}}(\Omega)$, where T'_μ is the tensor with coefficients

$$(T'_\mu)_{ij} = -\partial_i H_0 \partial_j h_\mu - \partial_j H_0 \partial_i h_\mu + \nabla H_0 \cdot \nabla h_\mu \delta_{ij},$$

and that T'_μ is divergence-free, implying that

$$\mu \nabla H_0 = 0. \qquad \square$$

Regularity issues. As in the case with magnetic field, if $\mu \in H^{-1}$, then $T \in L^1_{\text{loc}}(\Omega)$ and therefore $\operatorname{div} T = 0$ in the sense of distributions. Denoting by X the first column of T this means that $\operatorname{div} X = 0$ and $\operatorname{curl} X = 0$ in the sense of distributions. Thus X is smooth in Ω. But $|X| = |\nabla h_\mu|^2$ therefore $|\nabla h_\mu|^2$ is smooth. In particular $|\nabla h_\mu|$ is locally bounded in Ω hence h_μ is locally Lipschitz.

If μ is in $L^p(\Omega')$ for some subdomain Ω', then, exactly as in the case with magnetic field, the relation $\mu \nabla h_\mu = 0$ is true in Ω', and since $\mu = \Delta h_\mu$ and $\Delta h_\mu = 0$ a.e. on the set $\{\nabla h_\mu = 0\}$, we get $\mu = 0$ in Ω'. $\square$

Bibliographic notes on Chapter 13: In the case without magnetic field, and when the number of vortices (and the boundary condition) remain bounded independently of ε, the questions dealt with in this chapter were studied in the book of Bethuel–Brezis–Hélein [43]. The criticality condition for the limiting points and degrees was given in [43], following the derivation through matched asymptotics by Fife and Peletier [96]. Later work focused on the inverse problem, namely given points and degrees satisfying the condition, is it possible to find a corresponding sequence of solutions? We give relevant references in Chapter 14.

The results with magnetic field and with possibly unbounded numbers of vortices were obtained in [175], under more restrictive assumptions on the energy implying that the coefficients T_{ij} of the limiting tensors were in L^1 (the finite-part formulation was then not needed). The other cases dealt with here, in particular Case 1 of Theorem 13.2, and the finite-part formulation, are thus new extensions of these results.

Chapter 14

A Guide to the Literature

Our goal here is to give a brief overview of results on Ginzburg–Landau, and point towards suitable references (in thematic, rather than chronological or hierarchical order). We apologize for not being able to be completely exhaustive.

There have been a few review-type papers on Ginzburg–Landau that one can also refer to, notably [40, 155, 85, 68].

14.1 Ginzburg–Landau without Magnetic Field

14.1.1 Static Dimension 2 Case in a Simply Connected Domain

The first studies of that model, i.e., of the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

and its critical points, seem to date back to Elliott–Matano–Tang Qi [92] who proved that energy-minimizers have isolated zeroes, and to Fife and Peletier [96], who gave a formal justification of the "vanishing gradient property" for solutions.

The energy E_ε was then studied in details by Bethuel–Brezis–Hélein, in [42] for the case without vortices and in [43] for the case with vortices, both times with a fixed Dirichlet boundary data g of modulus one. They derived the "renormalized energy" (or the Γ-limit) of the problem:

$$W((a_1,d_1),\dots,(a_n,d_n)) = -\pi\sum_{i\neq j} d_i d_j \log|a_i-a_j| - \pi\sum_i d_i R(a_i) + \frac{1}{2}\int_{\partial\Omega} \Phi_0 \left(ig, \frac{\partial g}{\partial \tau}\right).$$

where Φ_0 is the solution with zero average on the boundary of

$$\begin{cases} \Delta\Phi_0 = 2\pi\sum_i d_i\delta_{a_i} & \text{in } \Omega \\ \dfrac{\partial\Phi_0}{\partial\nu} = (ig, \dfrac{\partial g}{\partial\tau}) & \text{on } \partial\Omega \end{cases}$$

and $R(x) = \Phi_0(x) - \sum_i d_i \log|x-a_i|$. Convergence of minimizers and critical points under the assumption $E_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$, and of their vortices, was established, with the derivation of the *renormalized energy* and of the "vanishing gradient property" presented here in Chapter 13. We sum up some of their results below:

Theorem 14.1. (Bethuel–Brezis–Hélein [43]). *Let Ω be a strictly star-shaped simply connected domain of $\mathbb{R}^2$ and $g:\partial\Omega\to\mathbb{S}^1$ a smooth map of degree $d>0$.*

If u_ε minimizes E_ε among maps with values g on $\partial\Omega$. Then, as $\varepsilon\to 0$, up to extraction of a subsequence, there exist d distinct points $a_1,\dots,a_d\in\Omega$ such that $u_\varepsilon\to u_*$ in $C^k_{loc}(\Omega\backslash\cup_i\{a_i\})$ where

1. *u_* is an $\mathbb{S}^1$-valued harmonic map from $\Omega\backslash\{a_1,\dots,a_d\}$ to $\mathbb{S}^1$ with $u_* = g$ on $\partial\Omega$ and with degree $d_i=1$ around each a_i.*

2. *$(a_1,\dots,a_d)$ is a minimizer of the renormalized energy W with $d_i = 1$.*

3. *$E_\varepsilon(u_\varepsilon)\ge \pi d|\log\varepsilon| + W(a_1,\dots,a_d) + d\gamma + o(1)$.*

If u_ε is a sequence of solutions with $u_\varepsilon = g$ on $\partial\Omega$ and $E_\varepsilon(u_\varepsilon)\le C|\log\varepsilon|$, then, as $\varepsilon\to 0$ and up to extraction of a subsequence, there exist distinct points $a_1,\dots,a_n\in\Omega$, and degrees $d_1,\dots,d_n$ with $\sum_{i=1}^n d_i = d$, such that $u_\varepsilon\to u_*$ in $C^k_{loc}(\Omega\backslash\cup_i\{a_i\})$ where u_* is a harmonic map from $\Omega\backslash\{a_1,\dots,a_n\}$ to $\mathbb{S}^1$ with $u_*=g$ on $\partial\Omega$ and with degree d_i around each a_i. Moreover $((a_1,d_1),\dots,(a_n,d_n))$ is a critical point of W (the d_i's being fixed) and satisfies the "vanishing gradient property."

Their starshapedness assumption on the domain was removed and replaced for minimizers by simple-connectedness by Struwe [189].

A large literature followed, which we review in thematic rather than chronological order. Note that all the results we mention below in this section without magnetic field are under the assumption that $E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|$, i.e., concern bounded (as $\varepsilon \to 0$) numbers of vortices, and that this is one of the main limitations to adapting them to the case with magnetic field.

14.1.2 Vortex Solutions in the Plane

The existence of radial vortex solutions in the plane, i.e., solutions of the form $f_n(r)e^{in\theta}$ in polar coordinates, where f_n satisfies a certain ODE, was established by Hervé and Hervé [111] via the study of the ODE (note that these solutions have infinite energy for $n \neq 0$). As we saw in Theorem 3.2, it was established by Mironescu [142] that the only solution of degree ± 1 at infinity is the radial one (up to translation). For general solutions in the plane, the quantization result $\int_\Omega (1-|u|^2)^2 = 2\pi d^2$ where d is the total degree, was established by Brezis–Merle–Rivière [61], see Theorem 3.4; other qualitative results were obtained by Sandier and Shafrir [165, 186].

It is not yet fully known whether there can exist nonradial vortex solutions in the plane. These solutions would have a finite number of vortices of degree d_i which would have to satisfy the relation (related to the result of [61] and the Pohozaev identity)

$$\sum_i d_i^2 = (\sum_i d_i)^2.$$

Ovchinnikov and Sigal conjectured the existence of such solutions (having some rotational symmetry) and gave heuristic arguments to support this statement in [147] (see also Open Problem 4 in Chapter 15).

14.1.3 Other Boundary Conditions

More general Dirichlet data (of modulus not equal to one and even possibly vanishing) were studied by André–Shafrir [26]. Neumann boundary conditions were also considered, see for example Spirn [188] for a derivation of the renormalized energy in that case.

14.1.4 Weighted Versions

Versions of the energy with different potential terms, or weighted versions, meant to include possible pinning effects, such as

$$\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(a(x) - |u|)^2}{2\varepsilon^2}$$

or

$$\frac{1}{2}\int_\Omega p(x)|\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

were studied by André–Shafrir [25], Hadiji–Beaulieu [33, 34], Du–Lin [86].

14.1.5 Construction of Solutions

Once the main result of [43] is known, namely that critical points/minimizers of E_ε have vortices which converge to critical points/minimizers of the renormalized energy, it is natural to examine the interesting inverse problem: given a critical point of the renormalized energy, can one find sequences of solutions of (1.3) whose vortices converge as $\varepsilon_n \to 0$ to these points? This has been solved under the restriction that vortices all be of degree ± 1; first for the case of local minimizers and min-max solutions by Lin [128] then more completely in the book by Pacard and Rivière [148] by a method of local inversion in weighted Hölder spaces, which also allowed them to establish a very nice uniqueness result, i.e., a one-to-one correspondance between solutions on the one hand, and critical points of the renormalized energy on the other hand, at least under this $d = \pm 1$ degree assumption. Another proof (via local inversion methods), which lifts the assumption of nondegeneracy of the renormalized energy, was recently given by Del Pino–Kowalczyk–Musso [82].

In the case of zero degree (or no vortices), a uniqueness result had been previously established by Ye and Zhou in [196].

Other unstable solutions were obtained by Almeida–Bethuel through topological methods [14].

14.1.6 Fine Behavior of the Solutions

The location and rate of convergence of the zeroes of solutions to the limiting vortices, was established by Comte–Mironescu [78] (results also follow from the study done in [148]). Also, the precise asymptotic expansion of the energy of (nonminimizing) solutions was established by Comte–Mironescu in [77, 79], through a minimality property of the solutions outside of their zero-set established in [79].

One may also mention a result of Bauman–Carlsson–Phillips [30] who proved that minimizing solutions with specific boundary data have a single zero.

14.1.7 Stability of the Solutions

In the case with Neumann boundary conditions, conditions on Ω for existence/nonexistence of nontrivial stable solutions (i.e., solutions with vortices) were given in [122, 123].

It was established in [183] that stable (resp. unstable) solutions of (1.3) have vortices which converge as $\varepsilon \to 0$ to stable (resp. unstable) critical points of the renormalized energy. A corollary of this result is that, for ε small enough, there does not exist a stable solution with vortices of (1.3) with Neumann boundary condition (in a simply connected domain), i.e., (1.3) with Neumann boundary condition cannot stabilize vortices. This had already been established but under the assumption that Ω is convex, and for every ε, by Jimbo and Sternberg in [125].

14.1.8 Jacobian Estimates

We saw in Chapter 6 that a crucial tool in the analysis of Ginzburg–Landau is the closeness between the Jacobian determinant $\mu = \mathrm{curl}(iu, \nabla u)$ and vortex densities $2\pi \sum_i d_i \delta_{a_i}$ measured in terms of the Ginzburg–Landau energy (see again Chapter 6 and [119]). A recent result of Jerrard and Spirn [120] gives improved estimates showing that the Jacobian can be made very close to some vortex density (where the vortices found this way are no longer the same ones as those given by the ball-construction method).

14.1.9 Dynamics

Heat-flow

Under the heat-flow for 2D Ginzburg–Landau, the limiting dynamical law of vortices, which is the gradient-flow of the renormalized energy

(up to collision time)

$$\frac{da_i}{dt} = -\frac{1}{\pi}\nabla_i W(a_1, \dots, a_n)$$

was proved, under a well-prepared data assumption, by Lin [129] and Jerrard–Soner [117], after slow motion had been observed by Rubinstein–Sternberg [161]. This result was retrieved through a more Γ-convergence or energy-based method in [174]. After the work of Bauman–Chen–Phillips–Sternberg [31], a few recent papers, by Bethuel–Orlandi–Smets [47, 48, 49] and by Serfaty [184], have extended the dynamical law passed collision and splitting times.

Schrödinger flow

This is also called the Gross–Pitaevskii equation, and is considered in superfluids, nonlinear optics and Bose–Einstein condensation. The limiting dynamical law of vortices

$$\frac{da_i}{dt} = -\frac{1}{\pi}\nabla_i^{\perp} W(a_1, \dots, a_n)$$

was established, still with well-prepared assumptions, by Colliander–Jerrard in [76] on a torus, and by Lin–Xin [134] in the whole plane. A recent result of Jerrard and Spirn [121] derives the same dynamical law for ε small but nonzero.

In the whole plane again, Bethuel and Saut [53] established the existence of some travelling wave solutions with vortices, as conjectured in the physics literature on the Gross–Pitaevskii equation, while Gravejat [104] proved the nonexistence of such solutions at supersonic speed.

Wave flow

In the case of the wave flow, the analogous limiting dynamical law was established by Lin in [130] and Jerrard in [114].

14.2 Higher Dimensions

14.2.1 Γ-Convergence Approach

In dimension 3, vortices become vortex-lines and in higher dimension, they become codimension 2 objects. The right way to capture them is to consider the analogue of the vorticity measure considered in this book (see Chapter 6.1), which is then a current, the Jacobian determinant of the function u. A result analogous to what is stated here in Theorem 6.1 was established by Jerrard–Soner in [119]. It served to prove similarly that these higher-dimensional vorticity-currents or weak Jacobians, $Ju = d(iu, du)$, are compact in the same weak norm, and that

$$\liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \geq \frac{1}{2}\|J\|$$

where $\|J\|$ is the total mass of the (rectifiable and integer-multiplicity) limiting Jacobian J; in other words, the Ginzburg–Landau functional is bounded below by $|\log \varepsilon|$ times half the mass of the limiting Jacobian, which is the mass (length, surface) of the limiting vortex lines or surfaces. A full Γ-convergence result (i.e., including the corresponding upper bound) was then established by Alberti–Baldo–Orlandi [12]. Some improvement of the lower bound, named "product-estimate", also used to estimate vortex velocities for vortex-dynamics, was established in [173].

14.2.2 Minimizers and Critical Points Approach

Even before the Γ-convergence approach, it was established that vortex-lines (in dimension 3 or higher) of minimizers should converge to minimal lines (or minimal connections): see Rivière [154], Sandier [167], Lin–Rivière [131]. It was also established that for critical points, they converge to stationary varifolds, see Lin–Rivière [132] and Bethuel–Brezis–Orlandi [44].

The case of the most general boundary data in 3D, i.e., boundary data in $H^{\frac{1}{2}}$ was examined in Bourgain–Brezis–Mironescu [57], in link with results on lifting of $\mathbb{S}^1$-valued maps in Sobolev spaces.

14.2.3 Inverse Problems

The inverse problem: given a curve which minimizes or is a critical point of length, construct solutions whose vortices converge to that curve, is beginning to be investigated. Montero–Sternberg–Ziemer [140] have proved that there exists such a locally minimizing solution (with Neumann boundary condition) if one starts from a straight line which is a local minimizer of length with endpoints on the boundary of the domain (hence the domain should be nonconvex), it was generalized to the case with magnetic field by Jerrard–Montero–Sternberg in [116]. By local inversion or Lyapounov–Schmidt type methods, Felmer–Kowalczyk–Del Pino [95] have established the existence of a critical point if one starts from a straight line whose endpoints are on the boundary, which is only a critical point of the length.

14.2.4 Dynamics

In dimension ≥ 3, the vortex-set of solutions of the Ginzburg–Landau heat-flow converges to a solution of mean curvature flow in the sense of Brakke (as for solutions to the Allen–Cahn equation). The first result in that direction was obtained in Lin–Rivière [133], and then a full proof was given by Bethuel–Orlandi–Smets [46].

As concerns the Schrödinger or Gross–Pitaevskii flow, of particular interest is the motion of a closed vortex loop. Such loops are expected to flow under binormal flow in the $\varepsilon \to 0$ limit of Gross–Pitaevskii. Results in that direction (but complete results only for the case of a travelling vortex circle) were obtained by Jerrard [115] and Bethuel–Orlandi–Smets [45]. Also, Chiron constructed travelling wave solutions, in particular helix-shaped ones [73, 74].

14.3 Ginzburg–Landau with Magnetic Field

14.3.1 Dependence on κ

As we saw in the phase diagram in Chapter 2, the qualitative behavior of the Ginzburg–Landau energy depends crucially on κ, the "Ginzburg–Landau parameter" which is a material constant.

The situation is most of the time divided into two cases: $\kappa < \frac{1}{\sqrt{2}}$ corresponding to type-I superconductivity, and $\kappa > \frac{1}{\sqrt{2}}$ corresponding to type-II superconductivity. The limiting situation $\kappa = \frac{1}{\sqrt{2}}$ is called the self-dual case. In that famous case, as observed by Bogomoln'yi, the functional can be rewritten into a sum of squares which can all be made equal to zero, and the Ginzburg–Landau equations decouple into a system of first order self-dual equations. For more on that case, refer to the book of Jaffe and Taubes [112].

The type of the superconductor is crucial for the behavior of vortices. Roughly speaking, when $\kappa < \frac{1}{\sqrt{2}}$, vortices (of same degree) would attract each other, hence they are not really observed but rather one observes interfaces (one-dimensional interfaces in 2D) between regions of superconducting phase $|u| \simeq 1$ and regions of normal phase $|u| \simeq 0$ (see for example [75] and references therein). In the self-dual case $\kappa = \frac{1}{\sqrt{2}}$, vortices do not interact and it was shown by Jaffe and Taubes in [112] that solutions with arbitrarily located vortices could be observed.

Then, for $\kappa > \frac{1}{\sqrt{2}}$ vortices of opposite sign attract and vortices of same sign repel, this is the regime where vortices and lattices of vortices are observed, as seen in this book. In this regime and in the context of the Yang–Mills–Higgs model on all $\mathbb{R}^2$, Rivière [156] showed that the unique (up to gauge-equivalence and reflection) minimizer is radially symmetric and of degree one.

However, the above classification is not completely accurate because it neglects size effects. The described classification with separation at the self-dual point $\kappa = \frac{1}{\sqrt{2}}$ corresponds rather to the situation for the whole plane (as in Abrikosov's study [1]) or large samples. In small samples, the scaling is such that the same behavior as for type-II superconductors (i.e., vortices) can be observed in superconductors with $\kappa < \frac{1}{\sqrt{2}}$, see for example Akkermans–Mallick [8] (and Schweigert–Peeters–Singha Deo [180] for corresponding numerical and experimental results) where branches of vortex-solutions such as in Chapter 11.1 are described. Another example of small size sample effect is described by Aftalion and Dancer in [3].

For a global picture, one may also refer to the paper by Aftalion and Du [4] which reviews the different regimes as a function of the parameters.

14.3.2 Vortex Solutions in the Plane

As we saw in Chapter 2, Section 2.5.1, the existence of the n-vortex, that is a finite-energy radial solution of the full Ginzburg–Landau equations (2.4) in $\mathbb{R}^2$, whose only zero is at the origin and of degree n, was first proved by Plohr [151, 152] and Berger–Chen [35]. Later on, their uniqueness (among radial solutions) was proved by Alama–Bronsard–Giorgi [10]. The stability of these vortex-solutions is crucially related to the type of the superconductor, as expected from the previous subsection. It was conjectured by Jaffe and Taubes and proved by Gustafson–Sigal [106] that

— for $|n| \leq 1$ the n-vortex is always stable
— for $|n| \geq 2$ the n-vortex is stable if $\kappa < \frac{1}{\sqrt{2}}$ and unstable if $\kappa > \frac{1}{\sqrt{2}}$.
The instability result had been previously established by Almeida–Bethuel–Guo [41] in the case of large enough κ. The stability of the degree 1 radial solution had also been established by Mironescu [141] (without magnetic field).

One can also search for possibly nonradial solutions in the plane, classifying them according to their homotopy class n, the homotopy class of a configuration being its topological degree at infinity, or its total degree. Jaffe and Taubes conjectured in [112] that for $\kappa > \frac{1}{\sqrt{2}}$, if $|n| > 1$ there are no finite action stable critical points in the n-homotopy class, and that for $n = 0, \pm 1$ the only stable critical point is the radial n-vortex solution described above. Rivière proved in [155] part of this in the strongly repulsive case of $\kappa \gg 1$. More precisely, he showed that for κ large enough, there is an energy-minimizer in the n-homotopy class if and only if $n = 0, \pm 1$, and that in that case it is the radial solution.

14.3.3 Static Two-Dimensional Model

Here we will restrict ourselves to the study of type-II superconductivity ($\kappa > \frac{1}{\sqrt{2}}$) and in particular, the London limit $\kappa \to +\infty$. There is abundant mathematical literature on 1-D solutions to the Ginzburg–Landau equations with studies of bifurcations, critical fields and asymptotics; we will not go into much detail, but refer to the works of Bolley–Helffer (for example [54]) and Aftalion–Troy [6].

Bethuel–Rivière [52] were the first to study vortices for the full Ginzburg–Landau model with magnetic-field, but with a Dirichlet type boundary condition (leading to a type of analysis similar to [43]). From now on, we restrict our attention to the standard full Ginzburg–Landau equations (GL), as studied in this book.

Critical fields and bifurcations

Here we will present the situation with decreasing applied fields.

Around H_{c_3}: As we already mentioned, above a third critical field H_{c_3}, the only solution is the (trivial) normal one $u \equiv 0$, $h \equiv h_{\text{ex}}$. Giorgi and Phillips have proved in [102] that this is the case for $h_{\text{ex}} \geq C\kappa^2$, which implies the upper bound $H_{c_3} \leq C\kappa^2$ for that constant C.

At H_{c_3}: Decreasing the applied field to H_{c_3}, a bifurcation from the normal solution of a branch of solutions with surface superconductivity occurs. The linear analysis of this bifurcation was first performed in the half-plane by De Gennes [80], then by Bauman–Phillips–Tang Qi [32] in the case of a disc (they thus analyze what is known as the "giant vortex" — a unique zero of u with very large degree); and for general domains, formally by Chapman [67], Bernoff–Sternberg [39], then rigorously by Lu and Pan [137], Del Pino–Felmer–Sternberg [81], Helffer–Morame [109], Helffer–Pan [108], see improved results in Fournais–Helffer [97, 98]. The nucleation of surface superconductivity takes place near the point of maximal curvature of the boundary, and the asymptotics for H_{c_3} is

Theorem 14.2.

$$H_{c_3} \sim \frac{\kappa^2}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} \max(curv(\partial\Omega))\kappa,$$

where β_0 is the smallest eigenvalue of a Schrödinger operator with magnetic field in the half-plane.

Between H_{c_2} and H_{c_3}: The behavior of energy minimizers for $H_{c_2} \le h_{\mathrm{ex}} \le H_{c_3}$ has been studied by Pan [149], who showed that, as known by physicists, minimizers present surface superconductivity which spreads to the whole boundary, with exponential decay of $|u|$ from the boundary of the domain. More qualitative results of this type were obtained by Almog in [20, 17, 19].

Around H_{c_2}: At H_{c_2}, one goes from surface superconductivity to bulk-superconductivity. It was established by Pan [149] that

$$H_{c_2} = \kappa^2.$$

Qualitative results on bulk-superconductivity below H_{c_2} were obtained in [172], establishing, in particular, how bulk-superconductivity increases (average) as h_{ex} is lowered immediately below H_{c_2}. Results of successive bifurcations and of almost periodic behavior were obtained recently by Almog [19, 21].

Regime $\log\kappa \ll h_{ex} \ll H_{c_2}$*:* In this situation, a uniform density of vortices fills the domain, as presented in Chapter 8 (and first established in [170]). This is where the Abrikosov lattice is expected.

Around H_{c_1}: The value of H_{c_1} and the behavior of minimizers around H_{c_1} were presented in details in this book, and previously established in the references quoted in Chapters 7, 11, 12.

Special solutions

Meissner solution:
The existence and stability of the Meissner solution (solution without vortices) up to the "superheating field" was studied by Bonnet–Chapman–Monneau [55], its uniqueness was also studied in [182]. The superheating field is defined precisely as the value of the applied field for which the Meissner solution loses its stability, and it is of order κ.

Vortex-solutions below the subcooling field:
The existence of branches of vortex-solutions was presented in Chapter 11. Previously, the existence of vortex-solutions for small applied fields $h_{\text{ex}} = O(1)$ had been established formally by Rubinstein [157, 158], and rigorously by Du and Lin [86]. The "subcooling field" is defined as the smallest applied field for which there exist stable vortex solutions. It is thus of order of a constant.

Radial solutions:
The radial degree-d (or d-vortex) solutions in a disc were studied by Sauvageot [177], for all values of κ. She established the existence and critical field for existence of these branches of solutions, as well as their stability and loss of stability through bifurcation of a branch of nonradial degree-d solutions.

Periodic solutions

We already mentioned the study of vortex solutions in the plane. In addition, periodic solutions naturally arise for the Ginzburg–Landau system, they are of critical importance to study the Abrikosov lattice. Since Abrikosov's original work [1], many periodic vortex solutions were exhibited, in general as bifurcating from the normal solution, in particular by Chapman [67] and Almog [16].

On the other hand, the study of the Ginzburg–Landau energy functional over periodic configurations (i.e., on a torus) was carried out by Dutour [89] and Aydi [28]. Dutour established a bifurcation diagram and studied in particular the bifurcation from the normal solution at $H_{c_2} = H_{c_3}$ (in the periodic case, there are no boundary effects). Aydi established that $H_{c_1} = \frac{1}{2}\log\kappa$ in the periodic setting, and studied the vorticity of minimizers for that order of applied fields, like in Chapter 7. He also constructed particular solutions which have vortices which concentrate on a finite number of lines.

14.3.4 Dimension Reduction

Chapman–Du–Gunzburger [70] have derived the two-dimensional limit of the 3D Ginzburg–Landau energy for thin films (when the thickness goes to 0). The limiting energy is like the 3D one but where the magnetic potential is prescribed, and the (possibly varying) thickness of the film results in a pinning term in the 2D model, see also Chapman–Héron [71] for a review of formal derivations. Jimbo and Morita [124] then proved that if there exists a nondegenerate solution of the two-dimensional problem, then the original 3D problem also has a local minimizer nearby.

Ginzburg–Landau in thin superconducting loops was also considered and Rubinstein and Schatzman (see [159] and references therein) derived the corresponding 1D model, with interpretation of the Little–Parks experiment. See also Rubinstein–Schatzman–Sternberg [160] for a model of thin loops including constrictions in order to model the Josephson effect.

14.3.5 Models with Pinning Terms

Various models containing weights were studied to take into account pinning effects: see Chapman–Héron [71] and the references therein, Aftalion–Sandier–Serfaty [5], Du–Ding [83], André–Bauman–Phillips [24] (who allowed zeroes of the pinning term). As mentioned just above, pinning terms arise naturally as a result of thin-film limits of the 3D Ginzburg–Landau model, they also serve to model impurities in the material. The analysis is also close to that done for the model without magnetic field and described above in Section 14.1.4.

14.3.6 Higher Dimensions

The full Ginzburg–Landau model in higher dimensions has not been studied as much as the two-dimensional one.

The main focus has been on the 3D analogue of the bifurcation study around H_{c_3}, on surface superconductivity and the influence of the geometry of the domain on its nucleation, see Pan [150], Almog [18], Helffer–Morame [110].

We already mentioned the inverse-type existence result of Jerrard–Montero–Sternberg [116]. More recently, Alama–Bronsard–Montero [11] derived a candidate for the first critical field in a ball in the presence of a uniform field, and constructed locally minimizing solutions with vortices. In the regime $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq C|\log \varepsilon|$, one may mention the result of Liu [136], which gives a curvature condition on the limiting vortex-lines of solutions, analogous to a result in Bethuel–Orlandi–Smets [45].

14.3.7 Dynamics

Here, again, the studies are quite similar in nature to the ones without magnetic field. For specific magnetic field results, see Du–Lin [86] and Spirn [187, 188] for the motion of a finite number of vortices in small applied fields, and Sandier–Serfaty [174] in large applied fields.

14.3.8 Mean-Field Models

A mean-field model describing the dynamics of a large number of vortices in the heat flow of Ginzburg–Landau was derived formally and through heuristic arguments by Chapman–Rubinstein–Schatzman [72] (see also similar work by E [90]). This model describes the evolution of vortices through an evolution-problem for the density-measure. Several mathematical papers were then interested in solving rigorously the evolution problem: see Schätzle–Styles [179], Lin–Zhang [135], Du–Zhang [87], Masmoudi–Zhang [139], Ambrosio–Serfaty [22].

The stationary case of the model is quite similar to the limiting conditions we obtained (rigorously) in Theorem 13.1. This stationary problem, in particular the regularity of the free-boundary (boundary of the support of the vorticity measure), was studied by Schätzle–Stoth [178], Bonnet–Monneau [56], Caffarelli–Salazar–Shagholian [66]. A higher dimensional-dynamical model was also proposed by Chapman [69], and later shown to be ill-posed by Richardson–Stoth [153].

14.4 Ginzburg–Landau in Nonsimply Connected Domains

In domain with holes, interesting phenomena of different qualitative nature occur, and many open problems remain. Due to the nontrivial topology, the order parameter can have a nonzero degree without vortices, in other words there can be vorticity (and permanent currents) without vortices.

For a review of such phenomena, we refer to the book edited by Berger and Rubinstein [36] completely devoted to the subject.

Let us mention that in the case with magnetic field, the existence and quantization of nontrivial solutions was studied by Rubinstein–Sternberg [162] and Almeida [13] (see also [15]). Also, Berger and Rubinstein [37] proved that in multiply-connected domains, the zero-set of the order parameter u can be of codimension 1, contrarily to the property of isolated zeroes for minimizers in simply-connected domains established by Elliott–Matano–Tang Qi. For a discussion on the Aharonov–Bohm effect see Helffer [107].

There is also some interesting dependence on the behavior of minimizers on the precise geometry of the domain, in particular on the conformal type in case of an annulus: see the results (obtained in the case without magnetic field) of Golovaty and Berlyand [103] (uniqueness of minimizer) and Berlyand and Mironescu [38].

Alama and Bronsard [9] have started to investigate the behavior of minimizers of the full energy G_ε under an applied magnetic field, i.e., the analogue of what is presented in this book but for nonsimply-connected domains. They establish, in particular, the existence and value of the first critical field for which vortices appear.

Chapter 15

Open Problems

1. Include the size of Ω as a parameter in the study, as it is done in Chapter 6.

2. Obtain, in Theorem 4.1, lower bounds which can be localized in each ball.

3. Obtain an analogue of Theorem 5.4 in the case where the number of vortices is unbounded (or the free-energy much bigger than $|\log \varepsilon|$).

4. Completely classify the solutions in $\mathbb{R}^2$ of $-\Delta u = u(1 - |u|^2)$ without further assumption. Prove or disprove the existence of the solutions conjectured by Ovchinnikov and Sigal in [147]. Classify the solutions of the Ginzburg–Landau equations with magnetic field in the whole $\mathbb{R}^2$. In both cases, describe the vortex-structures: is the zero set of u formed only of points or can it be formed of one-dimensional objects, such as lines? How can the points be arranged?

5. Obtain the lower order terms in the asymptotic expansions of the energy of solutions, i.e., find finer estimates of the energy, up to order $o(1)$. Deduce further information on the vortices of the solutions, in particular answer the question below. Also deduce that for energy-minimizers obtained in Chapter 7, there are really no vortices outside of the support of the limiting measure enclosed by the free-boundary.

6. As soon as the number of vortices diverges, our results on minimizers and local minimizers does not specify if each individual vortex

is of degree +1, it only says that the limiting vorticity measure is nonnegative, but there could be substructures of dipoles of vortices at smaller lengthscales.

In other words: can positive and negative degree vortices coexist in local minimizers of the energy? Are all vortices of degree +1?

7. Extract, from finer expansions of the energy, a minimization problem depending on the vortex points and leading to lattices, and be able to distinguish the least costly energetically between different shapes of lattices, and to identify them.

8. Obtain the existence of branches of stable n-vortex solutions up to superheating fields $O(\frac{1}{\varepsilon})$, i.e., extend the branches found in Chapter 11, relaxing the hypotheses of Definition 11.1, and determine the subcooling and superheating fields for which they lose their stability.

9. Does the Γ-convergence analysis for the energy with pinning terms as in [5], but relaxing the hypotheses on the points made there.

10. In the intermediate case studied in Chapter 9, treat the case where Λ is not reduced to one point, and the case where $D^2\xi_0(p)$ is degenerate. In particular, can the vortices be aligned and the limiting measure supported on a segment?

11. In Chapter 7, we established that limiting vorticity-measures of energy minimizers are minimizers of E_λ, and in Chapter 13 that limiting vorticity measures of critical points are somehow stationary points of E_λ. A first question is to describe more precisely what those limiting measures can look like, and how rigid the condition that they should be stationary is. Can the measures be negative? Can they be of changing sign?

12. In Chapter 9, we established similarly that in the intermediate case $n \ll h_{\text{ex}}$, limiting vorticity-measures of energy minimizers are minimizers of I. Can we prove then that, limiting vorticity measures of critical points (after blow-up) are stationary points of I or satisfy some criticality condition? What do these look like?

13. Similarly, in Chapter 11, we established that in the case $n = O(1)$, limiting vorticity measures of local energy minimizers are minimizers of w_n (or $R_{n,h_{\text{ex}}}$). Is it true that limiting vorticity-measures

of critical points are critical points of w_n? Describe what these look like and further push (than started in [105]) the study of this delicate discrete problem.

14. Analyze and classify the solutions of $\operatorname{div} T_\mu = 0$ in finite part (with the notation of Chapter 13). In particular, are the limiting vorticity measures supported only on points, lines or open sets?

15. Deduce extra conditions on the limiting vorticities for solutions as in Theorems 13.1 and 13.2, assuming, in addition, that they are *stable* solutions. Does this yields more regularity on the limiting measures?

16. In addition to what is proved in Chapter 13 (and Chapters 7, 9) obtain conditions on the limits of *blown-up sequences* of vorticity measures (at any scale $\gg \varepsilon$) of critical points.

17. Once these results are established, a wide class of problems is that of *inverse problems*: i.e., given limiting measures which are admissible, in the sense that $\operatorname{div} T_\mu = 0$ (cf. Theorems 13.1, 13.2 in Chapter 13), stationary for I, or critical for w_n, $R_{n,h_{\text{ex}}}$, etc, do there exist solutions of Ginzburg–Landau whose vorticities converge to those limits? Recall that some results of this type have been obtained on Ginzburg–Landau without magnetic field in dimensions 2 and 3, see Chapter 14.

    On Ginzburg–Landau with magnetic field, an example of such a result was given by Aydi [28] who constructed a sequence of solutions in a disc, whose limiting vorticity is a uniform measure supported on a circle. One can then ask whether for any simply connected domain Ω, there exist solutions whose vorticities converge to a measure supported on a closed curve. More generally, one can ask this for every admissible measure. Again can there be solutions with nonpositive/changing sign limiting measures?

18. As asked in Chapter 13, can there be critical points with a number of vortices much larger than h_{ex}?

19. In [173] we obtained results for the regime of h_{ex} approaching H_{c_2} from below, in which $h_{\text{ex}} = O(\frac{1}{\varepsilon^2})$. We proved, in particular, some upper and lower bounds on the energy, and the uniform repartition of energy in the domain at any scale $\gg \varepsilon$. No results were given

on the vortices however. A major challenge is to obtain a complete Γ-convergence in that regime, describe the vortices of minimizers, and if possible, see if they tend to arrange periodically in triangular lattices.

20. The dynamics of vortices for Ginzburg–Landau (with magnetic field) under the heat-flow was studied in [187] and [174]. It remains to study this for wider regime of fields, and for unbounded numbers of vortices (i.e., study the dynamics of the limiting vorticity-measures).

21. Extend the results with magnetic field to dimension 3. Even though there have been some results in that direction, much remains open.

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